R. L. WILDER, University of Michigan
ELLEN E. WILEY, Middlebury College
A. H. WILSON, Haverford College
ELIZABETH W. WILSON, Cambridge, Mass.
W. A. WILSON, Yale University
RUTH G. WOOD, Smith College
F. S. WOODS, Massachusetts Institute of Technology
FRANCES M. WRIGHT, Elmira College
J. W. YOUNG, Dartmouth College

The meeting at Providence was characterized by the very extensive plans formed for the entertainment of the guests. They were comfortably housed in the dormitories of Pembroke College of the University, and enjoyed the convenience of having their meals together in the dining room of Alumnae Hall. The parlors of this hall were continuously used as meeting places throughout each day and evening.

On Monday afternoon a large party was shown about Providence under the guidance of Mr. Norman M. Isham, an authority on Colonial architecture and furniture; numerous examples of New England architecture were visited. Monday evening President and Mrs. Clarence A. Barbour with Vice President and Mrs. Albert D. Mead received the guests in the parlors of Alumnae Hall, a welcoming speech was made by Doctor Barbour and delectable refreshments were afterward served. On Tuesday afternoon the ladies of the department of mathematics of Brown University served tea to the visiting guests.

Wednesday afternoon was observed as a half holiday, giving opportunity for the guests to go in automobiles or by bus to Newport, where they followed first the Shore Drive past beautiful estates, then the Cliff Walk of three and one half miles to the Casino, where a real shore dinner was served. On Thursday afternoon an opportunity was given to drive about the city, particularly to the beautiful Roger Williams Park and other scenic parts of Providence. The mathematicians were indebted to Miss Margaret Stillwell, custodian of the Annmary Brown Memorial, and to Mr. Lawrence Wroth, librarian of the John Carter Brown Library, for interesting special exhibits and many courtesies. Professor R. C. Archibald was frequently on duty throughout the week to show visitors the very remarkable mathematical library situated on the second floor of Wilson Hall, and cordial welcome was extended to the mathematicians to work in the library before or after the meetings; numerous visitors availed themselves of the opportunity to visit and to use the library. A resolution was adopted by rising vote at the close of the sessions of the Association expressing our appreciation of the fine hospitality shown by the administration and the department of mathematics of Brown University, and of the efficient service rendered by the committee on arrangements and the program committee.

The American Mathematical Society held its thirty-sixth summer meeting and fourteenth colloquium from Tuesday to Friday, with lectures by Professor Solomon Lefschetz of Princeton University on "Topology." A very gratifying increase in enrollment is to be noted as the colloquium lectures follow from summer to summer. Addresses were also given by Professor T. H. Hildebrandt of the University of Michigan and Professor J. D. Tamarkin of Brown Uni-

versity on topics in the general theory of linear operations and integral equations. Sessions for the reading of papers were held on Tuesday afternoon, Wednesday morning, and Thursday morning and afternoon.

The Mathematical Association held sessions on Monday afternoon and Tuesday morning, President Young presiding at the first session and Professor Kempner at the second session. An excellent program had been arranged by a committee consisting of Professors Marston Morse (chairman), A. A. Bennett, Lennie P. Copeland, and E. P. Lane. Abstracts of some of the papers are given, numbered in accordance with the numbers of the papers.

First Session of the Association

(1) "Heaviside operational calculus" by Dr. J. J. Smith, General Electric Company.

(2) "Random sampling" (illustrated) by Dr. W. A. Shewhart, Bell Telephone Laboratories.

(3) "Teaching and research in mathematics" by Professor W. A. Hurwitz, Cornell University.

1. The paper of Dr. Smith will appear in an early issue of the Monthly.

2. Dr. Shewhart's paper will also appear soon in the Monthly.

Second Session of the Association

(4) "Italian contributions to modern mathematics" by Professor Enrico Bompiani, Rome, Italy.

(5) "A mathematical theory of harmony and melody" (illustrated) by Professor G. D. Birkhoff, Harvard University.

(6) "The summer school for teachers of mathematics" by Professor H. P. Hammond, Director of summer school, Society for the Promotion of Engineering Education.

4. The Mathematical Association was greatly honored in having Professor Bompiani accept its invitation to speak at this summer meeting. He had been lecturing the past summer quarter at the University of Chicago, and is lecturing for a part of the present academic year in eastern universities. His presence at the meetings of the mathematical organizations brought with it much pleasure and he won many new friends by his geniality and pleasing personality. The paper will appear in a later issue of the Monthly.

5. Professor Birkhoff, assisted by Professor Marston Morse at the piano, developed a theory of harmony in which a comparison of chords and sequences of chords was effected by means of an analysis of the underlying aesthetic factors and application of a formula presented by him before the International Mathematical Congress of Bologna of 1928. The possibility of a similar treatment of simple melodies was also outlined. These conclusions will be contained in a book on the "Mathematical Elements in Art" which Professor Birkhoff is preparing.

6. The Summer School for Engineering Teachers, of the Society for the Promotion of Engineering Education, holds a particular interest this year for

teachers of mathematics, since it is probable that one of the sessions to be held during the summer of 1931 will be devoted to the study of methods of teaching mathematics to engineering students.

The Summer School had its inception in the general investigation of Engineering Education, conducted by the Society for the Promotion of Engineering Education in the years 1924 to 1929, inclusive. It is a continuing enterprise which has as its purpose concrete measures for the improvement of instruction in engineering. When it was established in 1927 there were no precedents for such an enterprise in higher education in America, and but one similar undertaking is now conducted, namely—the Summer School for Teachers of Engineering of Great Britain.

Since its establishment seven sessions of the Summer School have been held, as follows:

In 1927—Mechanics, at Cornell University and at the University of Wisconsin.

In 1928—Physics, at the Massachusetts Institute of Technology, and Electrical Engineering at the University of Pittsburgh.

In 1929—Mechanical Engineering at Purdue University.

In 1930—Engineering Drawing and Descriptive Geometry at the Carnegie Institute of Technology and Civil Engineering at Yale University.

From small beginnings, with forty teachers in attendance at each session the Summer School has grown to considerable proportions, the session of 1930 having been attended by a total of 190 teachers. Since its beginning, nearly 500 teachers have attended the sessions and the staffs have numbered nearly 200.

An idea of the nature of one of these sessions can be secured from a brief account of one of them. The program of the physics session was divided into a number of subdivisions:

No. 1. Educational principles and practices; for the study of the general principles of teaching. This division was under the direction of Professor Walter F. Dearborn, of the Graduate School of Education of Harvard University.

No. 2. Methods of teaching and subject matter of college courses in physics for engineers. This was the principal division of the program and was conducted under the direction of a full-time teaching staff, which included Professors W. S. Franklin, of Massachusetts Institute of Technology, O. M. Stewart, of the University of Missouri, and A. W. Duff, of Worcester Polytechnic Institute.

No. 3. The correlation of physics with other subjects of instruction, conducted under the leadership chiefly of teachers of engineering and of subjects related closely to physics.

No. 4. Applications of physical laws, a division in charge of practicing engineers and scientists.

No. 5. Advanced physics, a series of lectures conducted by prominent physicists from colleges and industries.

No. 6. Committee reports prepared by groups of members of the session on problems of importance in the teaching of physics.

No. 7. General lectures and recreation.

In accordance with the general plan, the Summer School sessions on different subjects or divisions of the curriculum are conducted each year in different institutions throughout the country. It is planned in this way, during a course of years, to cover all of the important divisions of engineering education and so to locate the sessions as to have them readily accessible to teachers in different parts of the country. It is notable, however, that all of the sessions thus far have been attended by men from widely separated localities. The session on civil engineering, in 1930, for example, numbered among its attendants representatives of thirty states and provinces of Canada and from one foreign country. Sixty different collegiate institutions were represented.

Members of the sessions are also widely representative of the different teaching grades and degrees of experiences, one of the sessions numbering, as an illustration, three deans, thirty professors, seventeen associate professors thirty assistant professors, and fourteen instructors.

The staffs of the sessions are recruited from among the foremost teachers, scientists and engineers of the country. They read, in fact, like a section from "Who's Who in Engineering Education."

The meetings are conducted through formal lectures, discussion periods, seminars, model teaching exercises, and committee work.

Of equal importance with the regular meetings are the informal gatherings of the teachers throughout the sessions for the discussion of their mutual problems. Arrangements are made at each session so that the entire group can live together for the period of the session, generally in one of the dormitories of the institution. Organized recreational features are provided.

When the Summer School was begun in 1927, the initiative for holding the sessions came from the Board and staff of the Society for the Promotion of Engineering Education, in charge of the work. For the past two or three years, however, the sessions have been held as the result of invitations received from groups or organizations of teachers. The American Mathematical Society and the Mathematical Association of America have thus expressed a desire to have one of the sessions devoted to the teaching of mathematics, and the Society for the Promotion of Engineering Education, through action of its Council, has approved the holding of such a session if the necessary arrangements can be made. The University of Minnesota has extended the invitation to hold this meeting at Minneapolis just prior to the meetings of the two mathematical societies in September, 1930. It is probable that the session will open on August 24 and close on September 5. The program will be devoted to educational principles, to the content and method of teaching mathematics, correlation with other subjects of the engineering curriculum, to advanced mathematics, and a number of other sub-divisions.

Applications to attend this session may be addressed to H. P. Hammond, Director of Summer School, Society for the Promotion of Engineering Education, 99 Livingston St., Brooklyn, N. Y.

Meeting of the Board of Trustees

Eight trustees were present at the meeting on Monday evening, Professors Hedrick, Huntington and Jackson being also present by invitation.

The trustees appointed Professor Lao G. Simons of Hunter College as a trustee to fill the vacancy occasioned by the death of Professor Cajori; she will serve until January 1932. A suitable resolution recognizing the Association's loss in Professor Cajori's death will be offered at the time of the annual meeting in December.

The following thirty-eight persons were elected to membership on applications duly certified:

To Individual Membership

C. E. Amos, B.Sc. (Denison). Grad. Asst., Ohio State Univ., Columbus, Ohio.

Brother Anselm, A.B. (Mt. St. Joseph's Coll.). Teacher, St. Xavier High School, Louisville, Ky.

W. E. Bobertz, M.S.E. (Michigan). Research Dept., Westinghouse Elec. and Mfg. Co., Wilkinsburg, Pa.

Anthony Braun, A.M. (Catholic Univ.). Monastery, Washington, D.C.

J. O. Chellevold, A.M. (Northwestern). Asst. Prof., St. Paul-Luther Coll., St. Paul, Minn.

H. V. Craig, Ph.D. (Wisconsin). Adj. Prof., Univ. of Texas, Austin, Tex.

H. B. Curry, Ph.D. (Göttingen). Asst. Prof., Pennsylvania State Coll., State College, Pa.

J. D. Dawson, M.S. (Ohio State). Prof., Antioch Coll., Yellow Springs, Ohio.

J. M. Feld, A.B., Chem. E. (Columbia). Instr., Columbia Coll., New York, N. Y.

Orrin Frink, Jr., Ph.D. (Columbia). Asso. Prof., Pennsylvania State Coll., State College Pa.

H. A. Garabedian, B.S. (Tufts). (F.A.S. 1930). Asst. Mathematician, John Hancock Mut. Life Ins. Co., Boston, Mass.

Grace T. Gunn, A.M. (Northwestern). Univ. of Omaha, Omaha, Nebr.

J. D. Hill, A.B. (California at L.A.). Asst., Univ. of California at Los Angeles, Los Angeles, Calif.

J. A. Hurry, A.M. (California). Prof. and Head of Dept., Western State Coll., Gunnison, Colo.

C. E. Kellam, A.M. (Chicago). Head of Dept., East Chicago High Schools, East Chicago, Ind.

V. V. Latshaw, Ph.D. (Indiana). Head of Dept., State Teachers Coll., Hays, Kans.

Ralph Lefever, A.M. (Nebraska). Strang, Nebr.

M. M. Lemme, A.M. (Indiana). Instr., Univ. of City of Toledo, Toledo, Ohio.

R. H. MacCullough, M.S. (Lafayette). Prof., Defiance Coll., Defiance, Ohio.

I. J. MacDonald, M.S. (St. Bonaventure's Coll.). Oratory Prep. School, Summit, N. J.

H. B. MacDougal, M.S. (Iowa). South Dakota State Coll., Brookings, S. Dak.

Edith A. McDougle, A.B. (Delaware). Instr., Women's Coll., Univ. of Delaware, Newark, Del.

James McGiffert, Ph.D. (Columbia). Prof., Grad. Dept., Rensselaer Polytech. Inst., Troy, N. Y.

H. E. Menke, A.M. (Ohio State). Asst. Prof., Otterbein Coll., Westerville, Ohio.

T. C. Morehouse, Mgr. College Dept., The Macmillan Co., New York, N. Y.

J. D. Novak, B.S. (Gettysburg). Univ. of Chicago, Chicago, Ill.

Rufus Oldenburger, M.S. (Chicago). Instr., Univ. of Michigan, Ann Arbor, Mich.

Oystein Ore, Ph.D. Prof., Yale Univ., New Haven, Conn.

Brother Paulinus, A.B. (Mt. St. Joseph's Coll.). Instr., St. Xavier High School, Louisville, Ky.

Mary S. Paxton, A.M. (Indiana). Teacher, S. S. High School, Fort Wayne, Ind.

James Pierpont, Ph.D. (Vienna). Prof., Yale Univ., New Haven, Conn.

Winnifred F. Pine, A.B. (Brown). Teacher, St. Margaret's School, Waterbury, Conn.

Ruth B. Rasmusen, M.S. (Chicago). Instr., South Dakota State Coll., Brookings, S. Dak.

C. G. REEN, M.S.E. (Michigan). Asst. Prof., Civil Eng., Gettysburg Coll., Gettysburg, Pa.
H. F. RIES, A.B. (Michigan). Asst. Actuary, Guaranty Life Ins. Co., Davenport, Iowa.
FLORA M. STREETMAN, A.M. (Rice Inst.). Houston, Tex.
H. A. TOOPS, Ph.D. (Columbia). Prof., Psychology, Ohio State Univ., Columbus, Ohio.
MARQUESS WALLACE, A.B. (Westminster Coll., Fulton, Mo.). Prin., and head of Dept., Missouri Milit. Acad., Mexico, Mo.

Aside from routine business of the trustees, a report was presented by Professor Dunham Jackson, chairman, of the Committee on Geometry, a joint committee of the Mathematical Association and the National Council of Teachers of Mathematics. This is to be sent to the trustees in mimeographed form with a view to action at the December meeting.

W. D. CAIRNS, *Secretary*

FLORIAN CAJORI

By LAO G. SIMONS, Hunter College of the City of New York

The death of Professor Florian Cajori occurred on August 14, 1930. He had undergone an operation toward the end of last year, but had satisfactorily recovered. The suddenness of the event increased the regret that followed upon its announcement.

Professor Cajori was born at St. Aignan near Thusis, Canton of Grisons, Switzerland, on February 28, 1859. He came to the United States in 1875 and in the course of time became a naturalized American citizen. His early education was obtained in Zillis, Thusis, then at the Kantonsschule in Chur (Grisons) and later at the State Normal School in Whitewater, Wisconsin. He received the degree of B.S. in 1883 and M.S. in 1886, both from the University of Wisconsin, (spending one year at this time as a student at Johns Hopkins University) and the degree of Ph.D. from Tulane University in 1894. The honorary degree of LL.D. was conferred upon him by Colorado College in 1913, that of Sc.D. by the University of Wisconsin in 1913, and the LL.D. again by the University of California in 1930.

Professor Cajori was associated professionally with Tulane University and Colorado College from 1885 to 1918 except for one year spent with the United States Bureau of Education. He was appointed professor of the history of mathematics at the University of California on July 1, 1918 and served in that capacity until July 1, 1929, when he retired with the title of professor of the history of mathematics, emeritus. He was a member of many scientific societies, among them: the Association of University Professors, the American Association for the Advancement of Science, the History of Science Society, the American Mathematical Society, the Mathematical Association of America, the National Council of Teachers of Mathematics, the Deutsche Mathematiker Vereinigung, and the Circolo Matematico di Palermo.

As already stated in the October issue of this Monthly, "he was a charter

member of the Mathematical Association of America and was one of an original group of four (later enlarged to twelve) representatives of mid-western universities and colleges who made possible the re-establishment of the American Mathematical Monthly on a sound financial basis." He served as the Association's president during 1917.

He was a member of the "National Conference of Ten" of the National Education Association "Committee of Ten", 1892, and a member of the "Geometry Syllabus Committee of Fifteen" 1910–1913, vice-president of the History of Science Society, 1924–1925, the first year of its existence. The last recognition that came to him from historical organizations was his election in 1929 as president of the Comité International d'Histoire des Sciences.

He was a familiar figure at meetings until recent years when he felt that the time consumed in this way must be spent on his own researches and writings. His last personal appearance on a program was at the History of Science Society meeting in Durham, North Carolina, last Christmas week when he read a paper on "A Century of American Geodesy."

Professor Cajori's special field was the history of mathematics. He was a pioneer in it in the United States and an indefatigable worker from as early as 1890, when his book on *The Teaching and History of Mathematics in the United States* was published by the U. S. Bureau of Education, until his death. A remarkable variety of articles and reviews in magazines flowed from his pen, the number from 1918 to 1930 exceeding one hundred-thirty. In addition, he published a number of books. His work entitled *History of Mathematics* appeared first in 1894 and *A History of Elementary Mathematics with Hints on Methods of Teaching* in 1896. (A later edition of the latter was translated into Japanese and published in Tokyo in 1929.) These were followed by *History of Physics*, 1899; Abschnitt XX in M. Cantor's *Vorlesungen über Geschichte der Mathematik*, 1908; *History of the Logarithmic Slide Rule*, 1909; *William Oughtred*, 1916; *History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*, 1918; *The Early Mathematical Sciences in North and South America*, 1928; *Mathematics in Liberal Education*, 1928; *Checquered Career of Ferdinand Rudolph Hassler*, 1929; *History of Mathematical Notations*, in two volumes, 1928–1929.

The following books were about ready for publication at the time of Professor Cajori's death and will be printed in the near future: *Newton's "Principia"* and *Newton's "Optics."*

Particularly valuable contributions by Professor Cajori are found in *Sir Isaac Newton, 1727–1927*, published in 1928 and *A Source Book in Mathematics*, 1930. Interspersed during these years are several elementary and college textbooks. The words on the memorial over the tomb of Sir Christopher Wren in St. Paul's Cathedral, London, "Lector, si monumentum requiris, circumspice", may well be transferred from stone to living words in the case of this historian of mathematics.

The ability of Professor Cajori to do continuous, concentrated work was

phenomenal. The painstaking effort of years went into his monumental work *A History of Mathematical Notations*. Such a result as this is the outcome of endless search, powers of observation and an almost miraculous recognition of possible sources of information. He was like a magnet in his chosen field working to draw items of interest to himself, and when he came to combine these items into the written work, his acute observations and deductions made his writings no mere recital of facts, valuable as this might have been. Instead there are the suggestions as to improvement along many lines which a study of history would provide.

Professor Cajori had a charm of personality that made him an endearing character to those with whom he came in contact. He was a fine type of gentleman, interesting to listen to and interesting to talk to. His influence was far-reaching extending to his students and colleagues, to his associates around the world in many lines of research, and to a wide circle of friends. That influence will continue to be felt through the presence of the man in his historical writings.

A GENERALIZED FOURIER SERIES REPRESENTATION OF A FUNCTION

By W. O. PENNELL, St. Louis, Mo.

Introduction

This paper describes a general method of obtaining, by operational analysis, a quasi Fourier series $S(x)$ representing a function $f(x)$ as follows:

$$S(x) = b^n f(x - na), \quad na < x < (n + 1)a, \tag{1}$$

where n takes on positive and negative integral values including zero, b is any real constant, and $f(x)$ is a function defined in the interval $0<x<a$. If unity is substituted for b the series reduces to the classical Fourier series for $f(x)$. $0<x<a$.

It will be noted that if $S_1(x)$ denote the Fourier series for $f(x)b^{-x/a}$ for the interval $0<x<a$ then

$$S_1(x) = f(x - na)b^{(-x-na)/a}, \quad na < x < (n + 1)a \tag{2}$$

where n as in (1) takes on positive and negative integral values including 0. By multiplying both sides of (2) by $b^{x/a}$ we get

$$S_1(x)b^{x/a} = b^n f(x - na)$$

which is the same as (1). It is obvious then that the series (1) may be obtained by multiplying by $b^{x/a}$ the classical Fourier series for $f(x)b^{-x/a}$, $0<x<a$.

Heaviside, in his *Electromagnetic Theory*, vol. 2, discussed operational methods of obtaining Fourier series for certain functions, but as far as the author is aware no general method of obtaining Fourier series for a function by operational means has been published.

Deduction of Formula

There is an operator ϵ^{-ap} known as the translation[1] operator which has the property of shifting the operand to the right or to the left a distance a, according to whether a is positive or negative. That is $\epsilon^{-ap}f(x)=f(x-a)$. This is true whether $f(x)$ (subject to certain restriction) is continuous or discontinuous. The proof of this action of the operator may be obtained for positive values of x rather easily from an application of Carson's integral equation[2] and for negative values of x from an application of a modified form of Carson's equation.

Now if we write

$$\epsilon^{ap}(\epsilon^{ap}+b)^{-1}f(x)P(x),$$

where $f(x)$ is the function for which we wish to obtain the generalized or quasi Fourier series and where $P(x)$ is a "pulse" defined as follows:

$$\begin{aligned} P(x) &= 1, \quad 0 < x < a, \\ &= \tfrac{1}{2}, \quad x = 0 \text{ and } \quad x = a, \\ &= 0, \quad x > a, \\ &= 0, \quad x < 0, \end{aligned}$$

we get by expanding the operator by long division:

$$\begin{aligned} (3)\ \epsilon^{ap}(\epsilon^{ap}+b)^{-1}f(x)P(x) &= [1 - b\epsilon^{-ap} + b^2\epsilon^{-2ap} - \cdots]f(x)P(x) \\ &= f(x)P(x) - bf(x-a)P(x-a) + b^2f(x-2a)P(x-2a) - \cdots \end{aligned}$$

which is evidently the right half of the series we wish or that portion of (1) for which $n \geqq 0$.

(3) can be expressed in operational form as follows:

$$f(x)P(x) = f(x)\ [1(x) - 1(x-a)], \quad \text{where } 1(x) \text{ is the unit}^3 \text{ function,}$$

that is where

$$\begin{aligned} 1(x) &= 1, \quad x > 0, \\ &= 0, \quad x < 0, \\ &= \tfrac{1}{2}, \quad x = 0, \end{aligned}$$

$$\begin{aligned} &= f(x)1(x) - \epsilon^{-ap}f(x+a)1(x), \\ &= [F(p) - \epsilon^{-ap}\psi(p)]1(x) \end{aligned}$$

where $F(p)$ and $\psi(p)$ are the operational equivalents[4] of $f(x)$ and $f(x+a)$ respectively.

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[1] This name was apparently first given to this operator by Oliver Heaviside in his *Electromagnetic Theory*, Vol. 2, page 111.

[2] See *Electric Circuit Theory and the Operational Calculus*, by J. C. Carson. (Published by the McGraw-Hill Book Co.)

[3] Many writers have defined the unit function as having all values between 0 and 1 for $x=0$. The definition given satisfies Dirichlet's condition and conforms with the results obtained at the points of discontinuity by a Fourier series.

[4] See *Heaviside's Operational Calculus*, by Berg (McGraw-Hill Co., 1929), *Electric Circuit*

So

$$\epsilon^{ap}(\epsilon^{ap}+b)^{-1}f(x)P(x) = [\epsilon^{ap}F(p)-\psi(p)](\epsilon^{ap}+b)^{-1}1(x). \tag{4}$$

The left hand half of the series or that part of (1) for which n is negative is evidently given by

$$\begin{aligned}-\epsilon^{ap}(b+\epsilon^{ap})^{-1}f(x)P(x) &= -b^{-1}f(x+a)P(x+a)\\ &\quad + b^{-2}f(x+2a)P(x+2a) - b^{-3}f(x+3a)P(x+3a)+\cdots\end{aligned} \tag{5}$$

(5) may be expressed in operational form as follows:

$$\begin{aligned}f(x)P(x) &= f(x)[1(-x+a)-1(-x)]\\ &= \epsilon^{-ap}f(x+a)1(-x)-f(x)1(-x)\\ &= [\epsilon^{-ap}\psi(p)-F(p)]1(-x).\end{aligned}$$

So

$$-\epsilon^{ap}(b+\epsilon^{ap})^{-1}f(x)P(x) = [\epsilon^{ap}F(p)-\psi(p)](b+\epsilon^{ap})^{-1}1(-x). \tag{6}$$

(4) is 0 if $x<0$ and represents the series of $x>0$.
(6) is 0 if $x>0$ and represents the series of $x<0$.
(4)+(6) represents the series for all values of x.
But

$$\begin{aligned}(4)+(6) &= [\epsilon^{ap}F(p)-\psi(p)](b+\epsilon^{ap})^{-1}[1(x)+1(-x)]\\ &= [\epsilon^{ap}F(p)-\psi(p)](b+\epsilon^{ap})^{-1},\end{aligned} \tag{7}$$

since $1(x)+1(-x)=1$.

If expression (7) is evaluated by the Heaviside expansion theorem or some other recognized operational method the generalized Fourier series is obtained.

It has not been rigorously demonstrated that the expansion theorem applies to an expression like $(\epsilon^{ap}+b)^{-1}$ or $(\epsilon^{ap}-b)^{-1}$. Regarding p as an algebraic quantity and by starting from the known[5] equality,

$$\frac{\epsilon^{-ap}-\epsilon^{ap}}{\epsilon^{-ap}+\epsilon^{ap}} = \sum_{n=1,2\ldots}^{\infty}\frac{-8ap}{(2n-1)^2\pi^2+4a^2p^2}$$

by rather simple algebraic manipulations we get

$$\frac{1}{\epsilon^{ap}+b} = \frac{1}{2b}-\frac{2}{b}\sum_{n=1,3,5\ldots}^{\infty}\frac{ap-\log_e b}{n^2\pi^2+(ap-\log_e b)^2},\quad b>0. \tag{8}$$

Theory & Operational Calculus, by J. R. Carson, (McGraw-Hill Co., 1926, *Operational Circuit Analysis*, by Bush (John Wiley and Sons, 1929), *The Practical Application of the Fourier Integral*, by Campbell (Bell System Technical Journal, October, 1928), *Generalization of Heaviside's Expansion Theorem*, by Pennell (Bell System Technical Journal, August, 1929), *Heaviside's Electric Circuit Theory*, by Cohen (McGraw-Hill Co., 1928), *Electromagnetic Theory*, by Oliver Heaviside, Vols. 2 & 3.

[5] Chrystal's *Algebra*, Vol. 2, page 338.

Similarly by starting with the known expansion of

$$(\epsilon^{ap}+\epsilon^{-ap})(\epsilon^{ap}-\epsilon^{-ap})^{-1},$$

it can be shown that

$$(9)\quad \frac{1}{\epsilon^{ap}-b}=-\frac{1}{2b}+\frac{1}{b(ap-\log_\epsilon b)}+\frac{2}{b}\sum_{n=1,2,3\cdots}^{\infty}\frac{ap-\log_\epsilon b}{4n^2\pi^2+(ap-\log_\epsilon b)^2},\quad b>0.$$

Applying the Heaviside expansion theorem to the partial fractions on the right of (8) and (9) we get

$$(10)\quad \frac{1}{\epsilon^{ap}+b}=\frac{1}{1+b}-\frac{1}{b}\sum_{n=1,3,5\cdots}^{\infty}\frac{2n\pi b^{x/a}\sin n\pi x/a+2(\log_\epsilon b)b^{x/a}\cos n\pi x/a}{\log_\epsilon^2 b+n^2\pi^2},\quad b>0$$

and

$$(11)\quad \frac{1}{\epsilon^{ap}-b}=\frac{1}{1-b}+\frac{b^{(x/a)-1}}{\log_\epsilon b}+\frac{2}{b}\sum_{n=1,2,3\cdots}^{\infty}\frac{b^{x/a}\log_\epsilon b\cos 2n\pi x/a+2n\pi b^{x/a}\sin 2n\pi x/a}{\log_\epsilon^2 b+4n^2\pi^2}.$$

If we apply the expansion theorem directly to $(\epsilon^{ap}+b)^{-1}$, $b>0$ we get the same result as by (10).

If, however, we apply the expansion theorem directly to $(\epsilon^{ap}-b)^{-1}$, $b>0$, we get the same result as given by (11) except that the term $b^{(x/a)-1}/\log_\epsilon b$ is missing.

The expansion theorem cannot then be applied directly to an expression $\epsilon^{ap}(\epsilon^{ap}-b)^{-1}$, $b>0$ unless the extra term is added to the result.

We will now obtain the generalized Fourier series for several functions.

Example 1: Find a series to represent $f(x)$ such that

$$\begin{aligned} f(x) &= 1, && 0<x<a,\\ &= 1-b, && a<x<2a,\\ &= 1-b+b^2, && 2a<x<3a, \quad \text{etc.}\end{aligned}$$

Here the pulse is an infinite one, namely $1(x)$. The operational expression is

$$\begin{aligned}\epsilon^{ap}(\epsilon^{ap}+b)^{-1}1(x) &= 1(x)-b\epsilon^{-ap}1(x)+b^2\epsilon^{-2ap}1(x)\cdots\\ &= 1(x)-b1(x-a)+b^21(x-2a)\cdots.\end{aligned}$$

To evaluate $\epsilon^{ap}1(x)(\epsilon^{ap}+b)^{-1}$ remember that the expansion theorem is given by

$$(12)\qquad \frac{Y(p)}{Z(p)}=\frac{Y(0)}{Z(0)}+\sum_{p_n}\frac{Y(p_n)}{p_nZ'(p_n)}\epsilon^{p_nx},$$

where $p_n=$ the roots of $Z(p)=0$. In this case $Y(p)=\epsilon^{ap}$, $pZ'p=ap\epsilon^{ap}$, and

$$\begin{aligned} p_n &= \text{the roots of } (\epsilon^{ap}=-b)\\ &= n\pi i/a+(\log_\epsilon b)/a,\\ n &= \pm 1,\ \pm 3,\cdots.\end{aligned}$$

So

$$f(x)=\frac{\epsilon^{ap}}{\epsilon^{ap}+b}=\frac{1}{1+b}+\sum\frac{\epsilon^{ap_n}}{ap_n\epsilon^{ap_n}}\,\epsilon^{p_nx}$$

$$=\frac{1}{1+b}+\sum_{\pm1,\pm3\ldots}^{\infty}\frac{\epsilon^{n\pi ix/a+x/a\log_\epsilon b}}{n\pi i+\log_\epsilon b}.$$

$$f(x)=\frac{1}{1+b}+2b^{x/a}\log_\epsilon b\sum_{n=1,3,5\ldots}^{\infty}\frac{\cos n\pi x/a}{n^2\pi^2+\log_\epsilon^2 b}+2\pi b^{x/a}\sum_{n=1,3,5\ldots}^{\infty}\frac{n\sin n\pi x/a}{n^2\pi^2+\log_\epsilon^2 b}. \tag{13}$$

In case $b=1$ (13) reduces to

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{1,3,5,\ldots}\frac{\sin n\pi x/a}{n}, \tag{14}$$

which is the classical Fourier series for

$$\begin{aligned} f(x)&=1, && 0<x<a,\\ &=0, && a<x<2a,\\ &=1, && 2a<x<3a, \quad \text{etc.}\end{aligned}$$

The values of (13) for $b=\frac{1}{2}$, $a=1$, and for the first three approximation curves have been calculated and plotted on Fig. I. The Gibbs effect will be seen at $x=0$, 1, 2, 3, etc.

The first wave between $x=0$ and $x=1$ is obviously the same as for the corresponding wave in the classical series where $b=1$. On Fig. I-A has been plotted and superimposed upon the generalized curve, the first wave of the Fourier curve, and it will be seen that the two waves are practically identical.

Example 2: Find the series for $f(x)$ when

$$\begin{aligned} f(x)&=\epsilon^x, && 0<x<c,\\ &=-b\epsilon^{x-c}, && c<x<2c,\\ &=+b^2\epsilon^{x-2c}, && 2c<x<3c, \quad \text{etc.}\end{aligned}$$

The pulse is

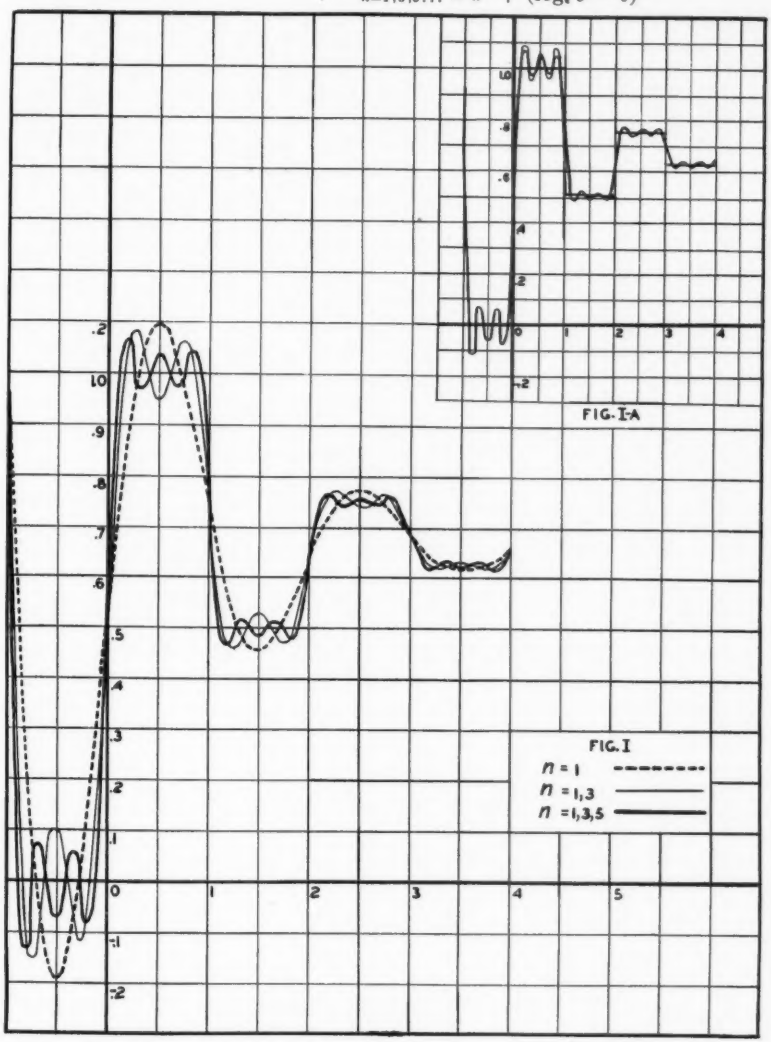
$$\begin{aligned} P(x)&=\epsilon^x[1(x)-1(x-c)]\\ &=p(p-1)^{-1}1(x)-\epsilon^{-cp}\epsilon^{x+c}1(x)\\ &=[p(p-1)^{-1}-p(p-1)^{-1}\epsilon^{-cp}\epsilon^c]1(x), \text{ since } \epsilon^x=p(p-1)^{-1}.\end{aligned}$$

So the operational expression for the series is

$$\frac{\epsilon^{cp}}{\epsilon^{cp}+b}\left[\frac{p}{p-1}-\frac{p\epsilon^{-cp}\epsilon^c}{p-1}\right]1(x)=\left[\frac{p}{(p-1)}\,\frac{\epsilon^{cp}}{\epsilon^{cp}+b}-\frac{p\epsilon^c}{(p-1)(\epsilon^{cp}+b)}\right]1(x). \tag{15}$$

Evaluating (15) by the expansion theorem we get

$$(16)\qquad f(x) = 2(b+\epsilon^c)(\log_\epsilon b - c)b^{(x/c)-1} \sum_{n=1,3,5\ldots}^{\infty} \frac{\cos(n\pi x/c)}{n^2\pi^2 + (\log_\epsilon b - c)^2} + 2(b+\epsilon^c)\pi b^{(x/c)-1} \sum_{n=1,3,5\ldots}^{\infty} \frac{n\sin(n\pi x/c)}{n^2\pi^2 + (\log_\epsilon b - c)^2}.$$



If $b=1$, (16) reduces to the classical Fourier series

$$(17)\quad f(x) = 2\pi(1+\epsilon^c) \sum_{n=1,3,5\ldots}^{\infty} \frac{n\sin(n\pi x/c)}{n^2\pi^2 + c^2} - 2c(1+\epsilon^c) \sum_{n=1,3,5\ldots}^{\infty} \frac{\cos(n\pi x/c)}{n^2\pi^2 + c^2},$$

where

$$f(x) = \epsilon^x, \quad 0 < x < c,$$
$$= -\epsilon^{x-c}, \quad c < x < 2c,$$
$$= \epsilon^{x-2c}, \quad 2c < x < 3c, \text{ etc.}$$

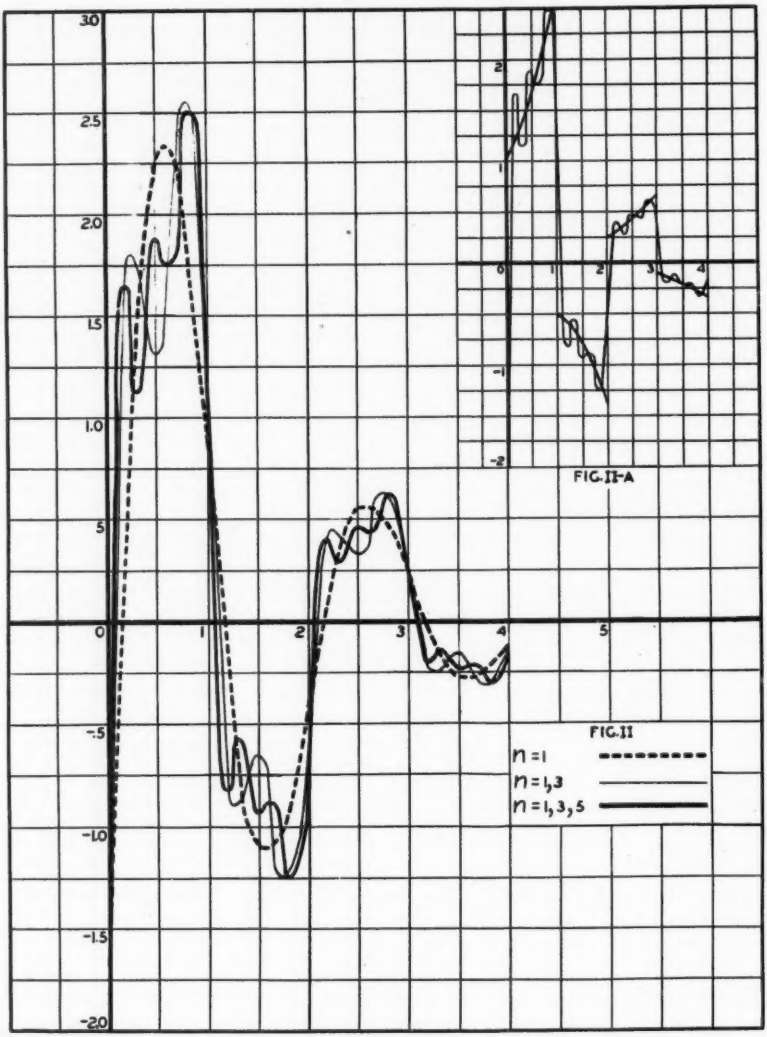


Fig. II shows the first 3 approximation curves. Here also the Gibbs effect is seen. Fig. II-A shows the third approximation curve superimposed upon the ultimate curve.

Example 3: Find the series for

$$f(x) = \cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$= b \cos x, \quad \frac{\pi}{2} < x < \frac{3\pi}{2},$$

$$= b^2 \cos x, \quad \frac{3\pi}{2} < x < \frac{5\pi}{2}, \quad \text{etc.}$$

The pulse is

$$\begin{aligned} P(x) &= (\cos x)1(x + \tfrac{1}{2}\pi) - 1(x - \tfrac{1}{2}\pi) \\ &= [\epsilon^{(\pi/2)p} \cos (x - \tfrac{1}{2}\pi) - \epsilon^{(-\pi/2)p} \cos (x + \tfrac{1}{2}\pi)]1(x) \\ &= \epsilon^{(\pi/2)p} (\sin x)1(x) + \epsilon^{(-\pi/2)p} (\sin x)1(x) \\ &= \epsilon^{(\pi/2)p} p(p^2 + 1)^{-1}1(x) + \epsilon^{(-\pi/2)p} p(p^2 + 1)^{-1}1(x) \\ &\qquad \text{since } \sin x = p(p^2+1)^{-1}. \end{aligned}$$

The operational expression for the wave is

$$(18) \quad \frac{\epsilon^{\pi p}}{\epsilon^{\pi p} + b} P(x) = \frac{p}{(p^2 + 1)} \frac{\epsilon^{(3\pi/2)p}}{(\epsilon^{\pi p} + b)} 1(x) + \frac{p}{(p^2 + 1)} \frac{\epsilon^{(\pi/2)p}}{(\epsilon^{\pi p} + b)} 1(x).$$

Solving (18) by the expansion theorem we get

$$(19) \quad f(x) = b^{x/\pi} \frac{(b^{1/2} - b^{-1/2})}{\pi} \sum_{n=1,3,5\cdots}^{\infty} \frac{2(1 - n^2 + \pi^{-2} \log_e{}^2 b) \cos n(x + \tfrac{1}{2}\pi) + 4n\pi^{-1} \log_e b \sin n(x + \tfrac{1}{2}\pi)}{[(n + 1)^2 + \pi^{-2} \log_e{}^2 b][(n - 1)^2 + \pi^{-2} \log_e{}^2 b]}.$$

The three first approximation curves are shown on Fig. III for the value $b = \frac{1}{2}$. Each half of the wave is a perfect sine curve, with the same wave length, but the amplitude of the 2nd half wave is one half that of the first. This is an entirely different curve from one in which the damping factor is exponential as for instance $\epsilon^{-ax} \cos x$.

When $b = 1$, (19) becomes equal to $\cos x$ which is the same result as obtained by Fourier series.

Example 4: Find the series for

$$\begin{aligned} f(x) &= x, && 0 < x < c, & f(x) &= b^{-1}(x + c), && -c < x < 0, \\ &= b(x - c), && c < x < 2c, & &= b^{-2}(x + 2c), && -c < x < -2c, \\ &= b^2(x - 2c), && 2c < x < 3c, & &= b^{-3}(x + 3c), && -2c < x < -3c, \\ &\text{etc.} & & & &\text{etc.} \end{aligned}$$

The pulse is

$$x[1(x) - 1(x - c)] = [p^{-1} - p^{-1}\epsilon^{-cp} - c\epsilon^{-cp}]1(x).$$

The operational equivalent is

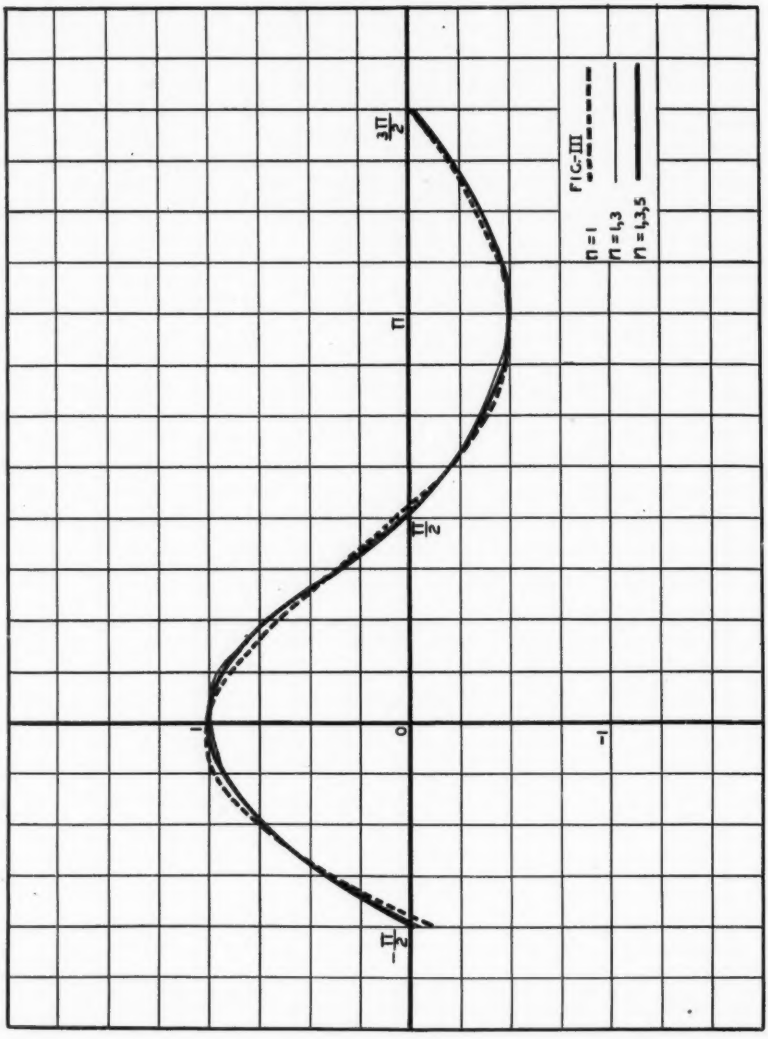
$$\epsilon^{cp}/(\epsilon^{cp} - b)^{-1}[p^{-1} - p^{-1}\epsilon^{-cp} - c\epsilon^{-cp}]1(x).$$

Solving this we get

$$(20)\quad f(x) = \frac{cb^{x/c}[b - 1 - \log_e b]}{b \log_e^2 b}$$

$$+ \frac{2(b-1)}{b} \sum_{n=1,2,3\cdots} \frac{cb^{x/c}(\log_e^2 b)\cos\left(\frac{2n\pi x}{c}\right) + 4cn\pi b^{x/c}(\log_e b)\sin\left(\frac{2n\pi x}{c}\right) - 4cn^2\pi^2 b^{x/c}\cos\left(\frac{2n\pi x}{c}\right)}{[\log_e^2 b + 4n^2\pi^2]^2}$$

$$+ \frac{2}{b} \sum_{n=1,2,3\cdots} \frac{-cb^{x/c}\log_e b\cos\left(\frac{2n\pi x}{c}\right) - 2cn\pi b^{x/c}\sin\left(\frac{2n\pi x}{c}\right)}{[\log_e^2 b + 4n^2\pi^2]}.$$

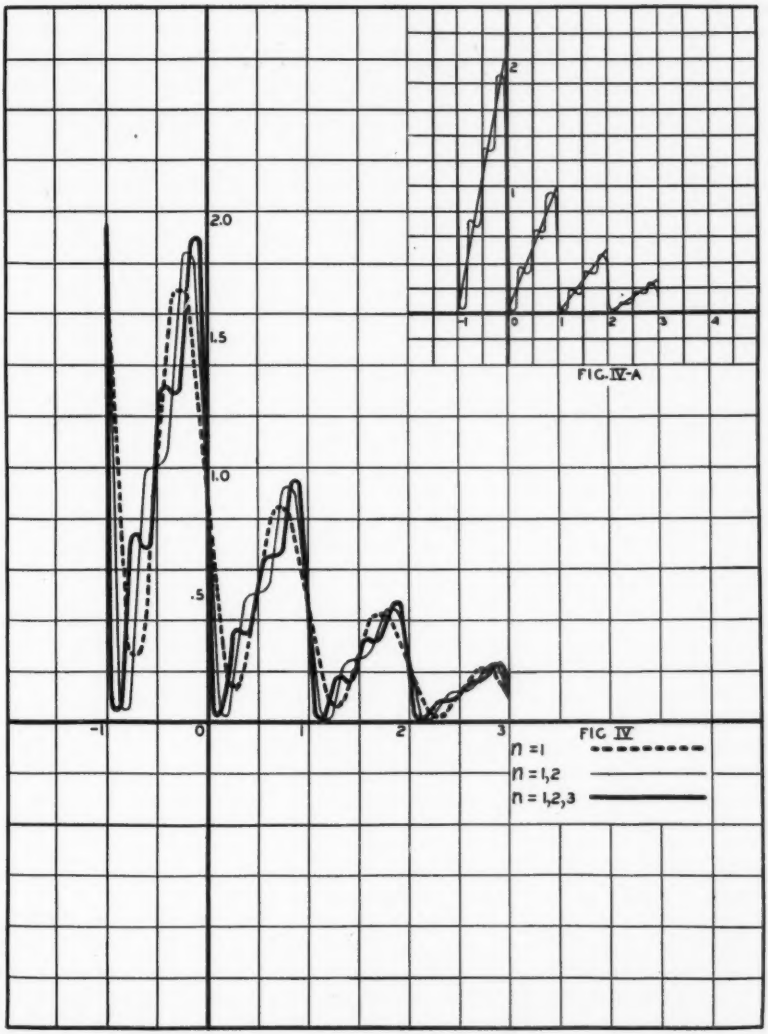


In (20) if $b=1$ we get

$$f(x) = \tfrac{1}{2}c - c\pi^{-1} \sum_{n=1,2,3\cdots} n^{-1} \sin(2n\pi x/c),$$

which is the Fourier expansion for

$$\begin{aligned} f(x) &= x, && 0 < x < c \\ &= x - c, && c < x < 2c, \\ &= x - 2c, && 2c < x < 3c. \end{aligned}$$



The curve represented by (20) when $c=1$ and $b=\frac{1}{2}$ and the first three approximation curves are shown on Figs. IV-A and IV.

Final Remarks

The operational process explained in this paper can be used, by starting with $b=1$, to derive directly, the classical Fourier series for a function. In general there is no saving in labor by deducing a Fourier series in this manner. For one accustomed to the operational methods it may be easier than the classical way, but the reverse is undoubtedly true for one used to the classical way.

The operational method does, naturally, open up the generalized series, which, while they can be obtained from the classical theory, as has been explained, are not generally known, at least the author has seen no reference to them.

In all the examples given the amount of shift in the translation operator is the same as the wave length of the pulse. This equality is not essential and by taking the amount of shift more or less than the wave length more complicated waves are produced and their sine and cosine series may be obtained.

It will be noted that the expansion is of the form $S(x)=A+Bb^{x/a}+Cb^{x/a}\sum^{\infty}[D_n \sin (n\pi x/a)+E_n \cos (n\pi x/a)]$, $n=1, 2, \cdots$, although in some cases some of the costants are zero.

The author wishes to acknowledge his indebtedness to Miss Elizabeth Harris who calculated and plotted the approximation curves, an undertaking of very considerable labor.

ON A CLASS OF FINITE SUMS[1]

By LEONARD CARLITZ, University of Pennsylvania

1. *Introduction.* The present paper grew out of an attempt to evaluate the sum of the n-th powers of the natural numbers from one to m. The expression most easily arrived at is in terms of the so-called Stirling[2] numbers and coincides with that as given, for example, by Schwatt.[3] Schwatt shows how useful the operator $(xd/dx)^n$ may be in this and like summations. Here we employ the generalized operator $(x^\lambda d^\lambda/dx^\lambda)^n$; this enables us to sum

$$\sum_{m=1}^{m}[m(m-1)\cdots(m-\lambda+1)]^n.$$

The sum involves a set of numbers that are a direct generalization of the Stirling numbers; it consists of $\lambda(n-1)+1$ terms (see equation (5)).

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[1] Presented to the American Mathematical Society, April 18, 1930.

[2] Stirling, *Methodus Differentialis*, p. 8.

[3] I. J. Schwatt, *Operations with Series* (1924), p. 86.

When λ is even it is possible to get a result involving but $\frac{1}{2}\lambda(n-1)+1$ terms. To effect this simplification use is made of a simple algebraic indentity (see equation (6)).

In the remainder of the paper, an attempt is made to consider abstractly arrays of numbers analogous to the arrays mentioned above. Explicit expressions for the elements of a large class of arrays are obtained; these arrays include the generalized Stirling numbers.

2. *The Operator* $(x^\lambda D^\lambda)^n$. Writing D for d/dx, we put

$$(1) \qquad (x^\lambda D^\lambda)^n = \sum_{s=1}^{(n-1)\lambda+1} a_{n,s}^{(\lambda)} x^{s+\lambda-1} D^{s+\lambda-1};$$

and we seek a recursion formula for $a_{n,s}^{(\lambda)}$. Applying $(x^\lambda D^\lambda)$ to both sides of this equality,

$$(x^\lambda D^\lambda)^{n+1} = \sum_s \sum_{i=0}^{\lambda} a_{n,s}^{(\lambda)} \binom{\lambda}{i} (s+\lambda-1)!^i x^{s+2\lambda-1-i} D^{s+2\lambda-1-i},$$

where

$$m!^k = m(m-1)\cdots(m-k+1).$$

From this

$$(2) \qquad a_{n+1,s}^{(\lambda)} = \sum_{i=0}^{\lambda} \binom{\lambda}{i} (s+i-1)!^i a_{n,s-\lambda+i}^{(\lambda)}.$$

(When $i=0$, $(s+i-1)!^i=1$.)

Equation (2) plus the "initial" conditions,

$$(3) \qquad \begin{cases} a_{1,1} = 1 \quad a_{1,s} = 0 \quad (s > 1), \\ a_{n,s} = 0 \quad \text{for} \quad s < 1, \end{cases}$$

define a triangular array of numbers. When $\lambda=1$ this is the array of Stirling numbers. When $\lambda=2$ we may easily calculate the first few elements, (n denotes the row, S the column.):

	1	2	3	4	5	6	7	8	9
1	1								
2	2	4	1						
3	4	32	38	12	1				
4	8	208	652	576	188	24	1		
5	16	1280	9080	16944	12052	3840	580	40	1

Let now $(x^\lambda D^\lambda)^n$ be applied to x^m. Then by (1),

$$(m!^\lambda)^n = \sum_{s=1}^{(n-1)\lambda+1} a_{n,s}^{(\lambda)} m!^{s+\lambda-1}. \tag{4}$$

But

$$\sum_{m=1}^{m} m!^k = \frac{(m+1)!^{k+1}}{k+1};$$

therefore, from (4),

$$\sum_{m=1}^{m} (m!^\lambda)^n = \sum_{s=1}^{(n-1)\lambda+1} a_{n,s}^{(\lambda)} \frac{(m+1)!^{s+\lambda}}{s+\lambda}. \tag{5}$$

3. *Summation of* (5) *for* λ *even.* We require the following identity:

$$m!^{2\lambda} = \sum_{i=0}^{\lambda} \Lambda_i (m+k+i)!^i (m-k-2\lambda)!^i \tag{6}$$

where

$$\Lambda_i = \Lambda_i(k) = \binom{\lambda}{i} (k+2\lambda)!^{\lambda-i} (k+\lambda)!^{\lambda-i}.$$

Proof: (6) may be proved by an induction on k. It obviously holds for $k = -\lambda$ and $1-\lambda$. Assuming then that (6) holds for k,

$$\begin{aligned} m!^{2\lambda} &= \sum_{i=0}^{\lambda} \Lambda_i(k)(m+k+i)!^i (m-k-2\lambda)!^i \\ &= \sum \Lambda_i(k) \cdot (m+k+1+i)!^i (m-k-1-2\lambda)!^i \\ &+ \sum i\Lambda_i(k) \cdot (2k+2\lambda+1+i)(m+k+i)!^{i-1} (m-k-1-2\lambda)!^{i-1} \\ &= \sum [\Lambda_i(k) + (i+1)\Lambda_{i+1}(k) \cdot (2k+2\lambda+2+i)] \\ &\quad \cdot (m+k+i+1)!^i (m-k-1-2\lambda)!^i. \end{aligned}$$

But

$$\Lambda_i(k) + (i+1)\Lambda_{i+1}(k) \cdot (2k+2\lambda+2+i) = \Lambda_i(k+1),$$

as may be verified without difficulty. From this

$$m!^{2\lambda} = \sum \Lambda_i(k+1)(m+k+1+i)!^i (m-k-1-2\lambda)!^i,$$

completing the induction.

Let us now put

$$(m!^{2\lambda})^n = \sum_{s=1}^{(n-1)\lambda+1} b_{n,s}^{(\lambda)} (m+s-1)!^{2(\lambda+s-1)}. \tag{7}$$

The possibility of this expansion will appear incidentally in getting a recursion formula for $b_{n,s}^{(\lambda)}$.

Assuming (7) and using (6), we have

$$(m!^{2\lambda})^{n+1} = \sum_{s=1}^{(n-1)\lambda+1} b_{n,s}^{(\lambda)}(m+s-1)!^{2(s+\lambda-1)} \cdot m!^{2\lambda}$$

$$= \sum_{i=0} b_{n,s}^{(\lambda)}(m+s-1)!^{2(s+\lambda-1)} \cdot \sum_{i=0}^{\lambda} \Lambda_i(s-1)(m+s+i-1)!^i (m-s-2\lambda+1)!^i$$

$$\left(\Lambda_i(s-1) = \binom{\lambda}{i}(s+2\lambda-1)!^{\lambda-i}(s+\lambda-1)!^{\lambda-i}\right)$$

$$= \sum_s \sum_i \binom{\lambda}{i}(s+2\lambda-1)!^{\lambda-i}(s+\lambda-1)!^{\lambda-i} b_{n,s}^{(\lambda)}(m+s+i-1)!^{2(\lambda+s+i-1)}.$$

Collecting the coefficient of $(m+s-1)!^{2(\lambda+s-1)}$

$$b_{n+1,s}^{(\lambda)} = \sum_{i=0}^{\lambda} \binom{\lambda}{i}(s+i-1)!^i (s+\lambda+i-1)!^i b_{n,s-\lambda+i}^{(\lambda)}. \tag{8}$$

This formula together with (3) (writing b in place of a) defines for each value of λ an array similar to those in §1. Using these numbers we may replace (5) by the following expression:

$$\sum_{m=1}^{m}(m!^{2\lambda})^n = \sum_{s=1}^{(n-1)\lambda+1} b_{n,s}^{(\lambda)} \frac{(m+s)!^{2\lambda+s-1}}{2\lambda+2s-1}. \tag{9}$$

It is perhaps unnecessary to remark that (7) might have been obtained by expanding the operator $(x^{2\lambda}D^{2\lambda})^n$ in terms of $(x^{2\lambda+s-1}D^{2(\lambda+s-1)}x^{s-1})$:

$$(x^{2\lambda}D^{2\lambda})^n = \sum_s b_{n,s}^{(\lambda)}(x^{2\lambda+s-1}D^{2(\lambda+s-1)}x^{s-1}). \tag{10}$$

However, the method used above seems simpler.

4. *Arrays in general. The operator E.*

Definition: We define the operator[4] E by

$$Ef(s) = f(s-1),$$

$f(s)$ being any function of the integral variable s.

Making use of E, some of our formulas may be put in much simpler shape. For example, (2) becomes

$$\begin{aligned} a_{n+1,s}^{(\lambda)} &= \sum_{i=0}^{\lambda} \binom{\lambda}{i}(s+i-1)!^i E^{\lambda-i} \cdot a_{n,s}^{(\lambda)} \\ &= (E+s)(E+s+1)\cdots(E+s+\lambda-1)\cdot a_{n,s}^{(\lambda)}. \end{aligned} \tag{11}$$

Similarly, (8) may be written:

[4] E and its inverse are familiar operators in the calculus of finite differences.

$$b_{n+1,s}^{(\lambda)} = \sum_{i=0}^{\lambda} \binom{\lambda}{i} (s+i-1)!^{i}(s+\lambda+i-1)!^{i} E^{\lambda-i} \cdot b_{n,s}^{(\lambda)}$$

$$= [E + s(s+\lambda)] \cdots [E + (s+\lambda-1)(s+2\lambda-1)] \cdot b_{n,s}^{(\lambda)} \tag{12}$$

$$= [E + s(s+2\lambda-1)] \cdots [E + (s+\lambda-1)(s+\lambda)] \cdot b_{n,s}^{(\lambda)}. \tag{13}$$

Note that the operators in (11) are permutable; those in (13) also are permutable, but those in (12) are not.

We shall now consider arrays of numbers $c_{n,s}^{(\lambda)}$ defined by (3) and

$$c_{n+1,s}^{(\lambda)} = \sum_{i=0}^{\lambda} \alpha_i^{(\lambda)}(s) E^{\lambda-i} \cdot c_{n,s}^{(\lambda)}. \tag{14}$$

The properties of these arrays will depend on the operator

$$\sum_i \alpha_i^{(\lambda)}(s) E^{\lambda-i}.$$

Let us take the simplest case $\lambda = 1$; we have then the operator

$$\beta(s)E + \alpha(s);$$

we shall take $\beta(s) \equiv 1$. We inquire when two such operators are permutable. Now

$$[E + \alpha(s)][E + \beta(s)] = E^2 + [\alpha(s) + \beta(s-1)]E + \alpha(s)\beta(s),$$

$$[E + \beta(s)][E + \alpha(s)] = E^2 + [\beta(s) + \alpha(s-1)]E + \beta(s)\alpha(s).$$

But from

$$\alpha(s) + \beta(s-1) = \beta(s) + \alpha(s-1)$$

follows

$$\alpha(s) - \beta(s) = \alpha(s-1) - \beta(s-1) = \mu,$$

a quantity independent of s.

Therefore, *a necessary and sufficient condition that* $E+\alpha(s)$ *and* $E+\beta(s)$ *be permutable is that* $\alpha(s)-\beta(s)$ *be independent of* s (cf. (11) and (13)). This result will be of use in §5.

We next take $\lambda = 2$, and consider

$$E^2 + A(s)E + B(s). \tag{15}$$

Let us attempt to exhibit it as a product

$$[E + \beta(s)][E + \alpha(s)].$$

Since $\alpha(s-1)+\beta(s) = A(s)$ and $\alpha(s)\beta(s) = B(s)$,

$$\alpha(s) = \frac{B(s)}{A(s) - \alpha(s-1)},$$

and we obtain the following expression of $\alpha(s)$ as a continued fraction:

$$\alpha(s) = \frac{B(s)}{A(s) -} \frac{B(s-1)}{A(s-1) -} \cdots \frac{B(1)}{A(1) - \nu}, \tag{16}$$

where $\nu = \alpha(0)$ is arbitrary. As (12) and (13) indicate, the decomposition of (15) is by no means unique; and indeed this is evident from (16). As an example, take

$$E^2 + 2sE + s(s+1);$$

then by (16),

$$\alpha(s) = \frac{s(s+1)}{2s -} \frac{(s-1)s}{2(s-1) -} \cdots \frac{1.2}{2\cdot 1 - \nu};$$

$$\frac{\alpha(s)}{s+1} = \frac{1}{2 -} \frac{1}{2 -} \cdots \frac{1}{2 - \nu}$$

$$= \frac{s - (s-1)\nu}{(s+1) - s\nu} = \frac{s(1-\nu) + \nu}{s(1-\nu) + 1},$$

so that

$$\alpha(s) = \frac{s(1-\nu) + \nu}{s(1-\nu) + 1}(s+1).$$

Since (15) can be expressed as a product in infinitely many ways it is natural to ask whether it is possible to pick ν in (16) so that the factors will coincide. In general this cannot be done; it is easy to show that a ν can be found only when

$$A(s) = [B(s)]^{1/2} + [B(s-1)]^{1/2}.$$

More generally, (15) can be expressed as a product of permutable operators only when

$$A(s) = [\mu + B(s)]^{1/2} + [\mu + B(s-1)]^{1/2}$$

for some constant μ.

5. *Explicit expressions for the elements of certain arrays.* We go back to the simple case $\lambda = 1$ of (14):

$$c_{n+1,s} = [E + \alpha(s)]c_{n,s},$$

Evidently

$$c_{n+1,s} = [E + \alpha(s)]^n c_{1,s}. \tag{17}$$

We shall now expand the right member of (17). If we put

$$(E + \alpha(s))^n = \sum_{i=0}^{n} A_i^{(n)}(s)E^i,$$

it is easily seen that

$$A_i^{(n)}(s) = \sum_{j=0}^{i} B_{ij}(s)\alpha^n(s-j). \tag{18}$$

Let i be fixed; from the definition, $A_i^{(i)}(s)=1$,

$$A_i^{(n)}(s) = 0 \quad \text{for} \quad 0 \leqq n \leqq i-1.$$

Solving the system (18) by Cramer's Rule,

$$B_{ij}(s) = \frac{(-1)^j}{\prod_{h=0}^{j-1}[\alpha(s-h)-\alpha(s-j)]\prod_{k=j+1}^{i}[\alpha(s-j)-\alpha(s-k)]}.$$

Therefore, using (18),

$$(19)\qquad E+\alpha(s))^n = \sum_{i=0}^{n}\left\{\sum_{j=0}^{i}\frac{(-1)^j\alpha^n(s-j)}{\prod_{h=0}^{j-1}[\alpha(s-h)-\alpha(s-j)]\prod_{k=j+1}^{i}[\alpha(s-j)-\alpha(s-k)]}\right\}E^i.$$

By (17) and (3),

$$(20)\qquad {}^{1}c_{n+1,s} = \sum_{j=1}^{s}\frac{(-1)^{s-j}\alpha^n(j)}{\prod_{h=1}^{j-1}[\alpha(j)-\alpha(h)]\prod_{k=j+1}^{s}[\alpha(k)-\alpha(j)]},$$

from which all the remaining formulas of this section are derived. We next take the array defined by two permutable operators $E+\alpha(s)$ and $E+\beta(s)$; $\beta(s)$ is then $=\alpha(s)+\mu$. Hence, starting with

$$\begin{aligned} c^{(2)}_{n+1,s} &= (E+\alpha(s))(E+\alpha(s)+\mu)c^{(2)}_{n,3} \\ &= \{(E+\alpha(s))(E+\alpha(s)+\mu)\}^n c^{(2)}_{1,s} \\ &= (E+\alpha(s))^n(E+\alpha(s)+\mu)^n c^{(2)}_{1,s} \\ &= \sum_{i=0}^{n}\binom{n}{i}\mu^{n-i}(E+\alpha(s))^{n+i}c^{(2)}_{1,s} \\ &= \sum_{i=0}^{n}\binom{n}{i}\mu^{n-i}\sum_{j=1}^{s}\frac{(-1)^{s-j}\alpha^{n+1}(j)}{\prod(\alpha(j)-\alpha(h))\prod(\alpha(k)-\alpha(j))}, \end{aligned}$$

we find that

$$(21)\qquad c^{(2)}_{n+1,s} = \sum_{j=1}^{s}\frac{(-1)^{s-j}\alpha^n(j)(\alpha(j)+\mu)^n}{\prod(\alpha(j)-\alpha(h))\prod(\alpha(k)-\alpha(j))}.$$

The close analogy between (20) and (21) indicates clearly the form of $c^{(\lambda)}_{n+1,s}$ for

$$(E+\alpha(s))(E+\alpha(s)+\mu)\cdots(E+\alpha(s)+\pi).$$

We shall now apply (20) and (21) to (11) and (13). For (11), $\lambda=1$, and we find

$$(22) \qquad a^{(1)}_{n,s} = \frac{1}{(s-1)!} \sum_{i=1}^{s} \binom{s-1}{s-i} (-1)^{s-i} i^{n-1}.$$

This may be put in the form

$$\frac{1}{s!} \sum_{i=1}^{s} (-1)^{s-i} \binom{s}{i} i^n,$$

which is the expression given, among other places, in Schwatt's[5] book. For general λ, (11) yields

$$(23) \quad a^{(\lambda)}_{n,s} = \frac{1}{(s-1)!} \sum_{i=1}^{s} (-1)^{s-i} \binom{s-1}{i-1} [i(i+1)\cdots(i+\lambda-1)]^{n-1}.$$

The treatment of (13) is somewhat more detailed; we find for $\lambda = 1$

$$(24) \qquad b^{(1)}_{n,s} = \frac{1}{(2s+1)!} \sum_{k=1}^{s} (-1)^{s-k} \binom{2s+1}{s-k} (2k+1)[k(k+1)]^n.$$

For $\lambda = 2$, we take $\alpha(s) = s(s+3)$, $\mu = 2$.

Then

$$b^{(2)}_{n,s} = \frac{1}{(2s+3)!} \sum_{k=1}^{s} (-1)^{s-k} \binom{2s+3}{s-k} (2k+3)[k(k+1)(k+2)(k+3)]^n.$$

For general λ we find

$$(25) \quad b^{(\lambda)}_{n,s} = \frac{1}{(2s+2\lambda-1)!} \sum_{k=1}^{s} (-1)^{s-k} \binom{2s+2\lambda-1}{s-k} (2k+2\lambda-1)[k\cdots(k+2\lambda-1)]^n.$$

It should be noticed that (21) can be obtained by the method of finite differences; this is apparently not the case with (25). At any rate the method used above enables one to easily obtain an explicit expression for the elements of any array corresponding to operators composed of permutable linear operators.

6. *A property of permutable operators and their associated arrays.*

Let $F = \alpha_0(s)E^i + \cdots + \alpha_i(s)$ *and* $G = \beta_0(s)E^j + \cdots + \beta_j(s)$ *be any two permutable operators. To* F *and* G *correspond the arrays* $f_{n,s}$, $g_{n,s}$; *to the product* $H = FG$ *corresponds the array* $h_{n,s}$. *Then the* $(2n-1)$st *row of the array that begins as* $f_{n,s}$ *for the first* n *rows and continues as* $g_{n,s}$ *is the* n*-th row of* $h_{n,s}$.

Proof: Let η_s be the element in the $(2n-1)$st row and s-th column of the compound array described above. Evidently

$$\eta_s = G^{n-1}F^{n-1}c_{1,s},$$

where of course $c_{1,s}$ is zero for $s \neq 1$ and one for $s = 1$. Now

$$h_{n,s} = H^{n-1}h_{1,s} = H^{n-1}c_{1,s}.$$

But, since $FG = GF$,

$$H^{n-1} = G^{n-1}F^{n-1},$$

which completes the proof.

[5] Loc. cit., p. 86. For other references, see Nielsen, *Nombres de Bernoulli*, p. 26.

A PROBLEM IN DIFFERENTIAL GEOMETRY

By B. F. KIMBALL, University of New Hampshire

One approves of using vectors in the study of differential geometry, but at times their seeming simplicity hides an essentially complicated state of affairs. Take, for instance, the vector equation:

$$\mathbf{r} = \mathbf{a}\cos t + \mathbf{b}\sin t + \mathbf{c}t\,, \tag{1}$$

where $\mathbf{r}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are position vectors. In a certain text, along with several simple problems, one is asked to prove that the above equation represents an elliptical helix. The above vector form at once shows that the curve lies on an elliptical cylinder, and that it winds around the cylinder, the various windings being at equal distances apart as measured along elements of the cylinder. But it is not, strictly speaking, a helix unless $\mathbf{c}$ has special directions, as determined by the relative lengths of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and in *that* case the curve does not make a constant angle with the *elements of the cylinder*. Perhaps the author of the text is using the term "helix" in a rather general sense.

It is interesting to note what the facts are concerning this curve. We use a dot over a letter to denote differentiation with respect to the parameter t. Let τ denote the torsion, k the curvature and s the length of the curve. The vector product is denoted by an $\times$ and the scalar product by a dot placed between the letters denoting the vectors. A parenthesis containing three vectors indicates the determinant of their vector components. A familiar formula gives:

$$\frac{\tau}{k} = \frac{(\dot{\mathbf{r}}\,\ddot{\mathbf{r}}\,\dddot{\mathbf{r}})}{k^3(\dot{\mathbf{r}}^2)^3}\,. \tag{2}$$

Now for the above curve

$$(\dot{\mathbf{r}}\,\ddot{\mathbf{r}}\,\dddot{\mathbf{r}}) = (\mathbf{abc})\,. \tag{3}$$

Thus a necessary condition that τ/k be constant is that the quantity $k^2(\dot{\mathbf{r}}^2)^2$ be constant. One finds that

$$k^2(\dot{\mathbf{r}}^2)^2 = \frac{[(\mathbf{a}\times\mathbf{b}) + (\mathbf{c}\times\ddot{\mathbf{r}})]^2}{\dot{\mathbf{r}}^2} \tag{4}$$

We wish to find conditions on $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ which make this a constant. We write

$$W[(\mathbf{a}\times\mathbf{b}) + (\mathbf{c}\times\ddot{\mathbf{r}})]^2 = \dot{\mathbf{r}}^2, \tag{5}$$

where W is constant. On expanding $\dot{\mathbf{r}}^2$ and $[(\mathbf{a}\times\mathbf{b})+(\mathbf{c}\times\ddot{\mathbf{r}})]^2$ one finds that each is equal to an expression in t of the form

$$A\cos 2t + B\sin 2t + C\cos t + D\sin t + E. \tag{6}$$

Thus in order that (5) be an identity in t, corresponding coefficients of (6) must be equal and one obtains the five conditions:

$$\frac{b^2 - a^2}{2} = \frac{(\mathbf{c} \times \mathbf{a})^2 - (\mathbf{c} \times \mathbf{b})^2}{2} W;$$

$$-\mathbf{a} \cdot \mathbf{b} = (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{c} \times \mathbf{b}) W;$$

$$-2(\mathbf{c} \cdot \mathbf{a}) = -2(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{b}) W; \tag{7}$$

$$+2(\mathbf{c} \cdot \mathbf{b}) = -2(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{a}) W;$$

$$c^2 + \frac{a^2 + b^2}{2} = \left[(\mathbf{a} \times \mathbf{b})^2 + \frac{(\mathbf{c} \times \mathbf{a})^2 + (\mathbf{c} \times \mathbf{b})^2}{2} \right] W.$$

We define three angles α, β, and γ to satisfy the relations

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \alpha, \quad \mathbf{a} \cdot \mathbf{c} = ac \cos \beta, \quad \mathbf{b} \cdot \mathbf{c} = bc \cos \gamma,$$

where the italic letters denote the lengths of the vectors. Thus solving the 2nd, 3rd, and 4th equations for the cosines of these angles one obtains;

$$\cos^2 \alpha = \frac{(Wa^2 - 1)(Wb^2 - 1)}{W^2 a^2 b^2}, \quad \cos^2 \beta = \frac{(Wa^2 - 1)(Wc^2 + 1)}{W^2 a^2 c^2}, \tag{8}$$

$$\cos^2 \gamma = \frac{(Wb^2 - 1)(Wc^2 + 1)}{W^2 b^2 c^2}.$$

Now using the values of the sines obtained from the above relations it is found that the first and last of equations (7) are identically satisfied. Thus the equations (8) are necessary conditions that the curve (1) be a helix. Without loss of generality in the discussion, one can take $\mathbf{a}$ and $\mathbf{b}$ as giving the major and minor axes of the ellipse defined by the first two terms of (1). This allows two possible choices of W. If one takes $W = 1/a^2$ one finds that $\cos^2 \gamma$ will be negative. This leads to the conditions

$$W = 1/b^2, \quad \cos^2 \alpha = 0 \tag{9}$$

and

$$\cos^2 \gamma = 0, \quad \cos^2 \beta = \frac{(a^2 - b^2)(b^2 + c^2)}{a^2 c^2}, \quad \sin^2 \beta = \frac{b^2(b^2 + c^2 - a^2)}{a^2 c^2}, \tag{10}$$

which show that in any case $\mathbf{c}$ must be perpendicular to the minor axis of the directrix ellipse.

Under (9) and (10) we have

$$k = \frac{b}{\dot{r}^2}, \quad \tau = \frac{(\mathbf{abc})}{b^2 \dot{r}^2}, \qquad (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})^2 = b^2 \dot{r}^2. \tag{11}$$

Let $\boldsymbol{d}$ denote the fixed direction, if such exists, with which the helix makes a constant angle. From the theory of the helix we have

$$\boldsymbol{d} \text{ parallel to } \tau \boldsymbol{t} + k\boldsymbol{b}$$

or

$$\boldsymbol{d} \text{ parallel to } \frac{(\boldsymbol{abc})}{b^2\dot{\boldsymbol{r}}^2}\,\frac{\dot{\boldsymbol{r}}}{\sqrt{\dot{\boldsymbol{r}}^2}} + \frac{b}{\dot{\boldsymbol{r}}^2}\,\frac{\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}}{|\,\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}\,|}. \tag{12}$$

Using (11) one can write this vector as

$$\frac{1}{b^2}\,\frac{1}{\dot{\boldsymbol{r}}^2}\,\frac{1}{\sqrt{\dot{\boldsymbol{r}}^2}}[(\boldsymbol{abc})\dot{\boldsymbol{r}} + b^2(\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}})].$$

We set

$$\boldsymbol{d} = (\boldsymbol{abc})\dot{\boldsymbol{r}} + b^2(\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}),$$

and expanding in terms of t and noting that

$$b^2(\boldsymbol{c}\times\boldsymbol{a}) = (\boldsymbol{abc})\boldsymbol{b}$$

because of the conditions (10), we find that

$$\boldsymbol{d} = [(\boldsymbol{abc})\boldsymbol{c} + b^2(\boldsymbol{a}\times\boldsymbol{b})] + [b^2(\boldsymbol{b}\times\boldsymbol{c}) - (\boldsymbol{abc})\boldsymbol{a}]\sin t.$$

Now the two brackets can be shown to represent parallel vectors under the choice of β given above.

Thus under the special conditions (10) the curve is a helix which makes a constant angle with the vector

$$(\boldsymbol{abc})\boldsymbol{c} + b^2(\boldsymbol{a}\times\boldsymbol{b}).$$

The cotangent of that angle will be $(\boldsymbol{abc})/b^3$.

EXTENSION OF THE CONCEPT OF GROUP OF ISOMORPHISMS

By G. A. MILLER, University of Illinois

It is well known that the group of isomorphisms, or the group of automorphisms, of a given group G is one of the most important concepts of group theory. Hence it may be desirable to note here a certain extension of this concept since this may throw some new light on its real nature. The group of isomorphisms of G may be constructed as follows: Write G in the form of a regular substitution group and construct the group formed by all the substitutions on the letters of G which satisfy the conditions that they transform G into itself and also omit a fixed letter of G. The substitution group thus formed is simply isomorphic with the group of isomorphisms of G when G is regarded as an abstract

group. When G is regarded as a regular substitution group the given substitution group may be regarded as the actual group of isomorphisms of G. The substitutions and the subgroups of G which are transformed into themselves under this group of isomorphisms are called characteristic substitutions and characteristic subgroups respectively of G. The symmetric group on the letters of G may contain many regular substitution groups which are simply isomorphic with G. In abstract group theory it is customary to regard all simply isomorphic groups as the same group and this is also done in the enumeration of the regular substitution groups. It is obvious that the symmetric group on all except one of the letters of G transforms G into all of its conjugates on its own letters and involves no two substitutions which transform all the substitutions of G in the same way. Hence such a symmetric group may be regarded as the *symmetric group of isomorphisms of* G, and to distinguish it from the group of isomorphisms of G noted in the preceding paragraph this may be called the *holomorph group of isomorphisms of* G. It may be noted that there are exactly g substitutions in the symmetric group on the letters of G which transform G according to the same substitution in each of these groups of isomorphisms, g being the order of G, and that at least one of these is of the same order as the isomorphism in question.

It is evident that under the symmetric group of isomorphisms of G there are no characteristic substitutions and no characteristic subgroups of G besides the identity except in the trivial case when the order of G is 2, and that these two definitions of the group of isomorphisms coincide when and only when G is one of the following four groups: The identity, the group of order 2, the group of order 3, and the non-cyclic group of order 4. All the groups of the same order have the same symmetric group of isomorphisms but they usually have different holomorph groups of isomorphisms. The quotient obtained by dividing the order of the symmetric group of isomorphisms by the order of the holomorph group of isomorphisms is equal to the number of the conjugates of G under the symmetric group on its letters. Characteristic substitutions and characteristic subgroups correspond to such substitutions and to such subgroups of the corresponding group under the symmetric group of isomorphisms but they do not necessarily correspond to themselves under the isomorphisms of this group.

Under the holomorph definition of the group of isomorphisms only the possible interchanges of the operators in the possible automorphisms of a group are considered while according to the extended definition isomorphisms with simply isomorphic groups are also considered. Since all simply isomorphic groups are frequently regarded as the same group and all such groups may be supposed to be represented as regular substitution groups on the same set of letters whose number is equal to their common order, it seems desirable to note the effect of such an extension of the notion of a given group on the concept of its group of isomorphisms. It should however be observed that the ordinary, or holomorph, definition of the group of isomorphisms seems to be the more useful, and should

always be understood unless the contrary is explicitly stated. In closing it may be added that the present note has close contact with the article published in this Monthly, volume 29(1922), page 319, and may be regarded as an extension thereof along certain lines.

POSSIBLE TYPES OF MULTIPLICATION OF SERIES

By E. T. BELL, California Institute of Technology

1. Throughout the theory of numbers there is a sharp and generally impassable break between additive and multiplicative properties. For example, in order to devise a definition of addition for ideals, it seems to be necessary to revise the existing theory, which is purely multiplicative, ab initio. Again, in rational arithmetic, some of the most intractable questions are those combining both additive and multiplicative properties, for example Goldbach's hypothesis that every even integer is the sum of two primes. For additive rational arithmetic the natural analytical tool is the power series; for multiplicative, the Dirichlet. It is conceivable but not probable that other types of arithmetic may exist; namely, sections of arithmetic in which the laws of algebra are preserved in so far as they are relevant, which are neither multiplicative nor additive only. In trying to disprove such existence, I was led to certain simple functional equations in two variables, one of which is restricted to be an integer $\geqq 0$; to settle the question indicated above it is sufficient to prove that the obvious solutions of these equations are the general solutions. Possibly these equations are familiar to some reader of this Monthly who will supply the desired proof.

2. Let x be a parameter, n an integer $\geqq 0$, and let a_n be independent of x. Let $a_n, f(x,n)$ be defined and single-valued for all values of x, n considered. Further let $a_n, f(x,n)(n=0,1, \ldots)$ be such that

$$2.1 \qquad A(x) \equiv \sum_{n=0}^{\infty} a_n f(x, n)$$

is uniquely defined and has a meaning such that the product of two functions of the same kind as $A(x)$ is a function of the same kind:

$$2.2 \qquad A(x) \cdot B(x) = C(x),$$

where

$$B(x) \equiv \sum_{n=0}^{\infty} b_n f(x, n), \quad C(x) \equiv \sum_{n=0}^{\infty} c_n f(x, n).$$

Further suppose that the terms in the product on the left of 2.2 can be rearranged in any way after multiplication without affecting the meaning of $A(x)B(x)$.

The multiplication in 2.2 will be possible if $f(x,n)$ is such that

$$2.3 \qquad f(x, n_1)f(x, n_2) = f(x, \phi(n_1, n_2)),$$

where $\phi(n_1, n_2)$ is an integer $\geqq 0$ whenever n_1, n_2 are integers $\geqq 0$. From the hypotheses it follows that $\phi(n_1, n_2)$ is a symmetric function of n_1, n_2,

$$2.4 \qquad \phi(n_1, n_2) = \phi(n_2, n_1).$$

It follows also from the hypotheses that the multiplication defined by 2.2 is associative. Hence, and by 2.4,

$$2.5 \qquad \phi(\phi(n_1, n_2), n_3) = \phi(\phi(n_3, n_2), n_1)$$

for every triple of integers $n_1,n_2,n_3 \geqq 0$. If for the moment we drop the restriction that n_1, n_2, n_3 be integers, and consider only polynomials $\phi(n_1, n_2)$, it follows by a simple argument that ϕ is of the first degree in n_1, n_2 separately, and that there are two essentially distinct solutions of 2.4, 2.5; namely,

$$2.6 \qquad \phi(n_1, n_2) = n_1 + n_2 + c;\ \phi(n_1, n_2) = cn_1n_2,$$

in which, if $\phi(n_1,n_2)$ is to be a positive integer when n_1,n_2 are positive integers, c is an arbitrary constant positive integer (independent of n_1,n_2).

3. The functional equations to be solved are therefore, from 2.6, 2.3,

$$3.1 \qquad f(x, n_1)f(x, n_2) = f(x, n_1 + n_2 + c),$$

$$3.2 \qquad f(x, n_1)f(x, n_2) = f(x, cn_1n_2).$$

A solution of 3.1 is $f(x, n) = [F(x)]^{n+c}$; a solution of 3.2 is $f(x, n) = (c \cdot n)^{F(x)}$; where, in either, $F(x)$ is an arbitrary function of x, arbitrary to the extent demanded by the definitions concerning 2.1.

Hence, taking a new variable $y = F(x)$, we see that 3.1, 3.2 define respectively a power series in y and a Dirichlet series in y, and that 2.2 is in the respective cases Cauchy multiplication or Dirichlet multiplication of series (for $c = 0$, $c = 1$ respectively).

What are the general solutions of 3.1, 3.2? It need not be assumed that $f(x,n)$ is a continuous function of x.

I have since found a much more general solution of 3.2 which, however, does not seem to settle the question of complete generality.

SOLUTION OF EQUATIONS BY ADDITION-SUBTRACTION LOGARITHMS

By P. H. NYGAARD, North Central High School, Spokane, Wash.

By the use of addition-subtraction logarithms (also called Gaussian logarithms) it is possible to find the logarithm of the sum or the difference of two numbers directly from their logarithms. The tables used for this purpose contain what are known as A, B, and C logarithms. All cases of addition can be done by means of the A and B logarithms. Substraction problems require either the A and B or the B and C combination depending upon how large a difference exists between the logarithms of the two numbers. The following discussion is based upon the addition-subtraction logarithms as found in Jones's *Logarithmic Tables*.[1]

The purpose of this paper is to present a method of approximating the values of the roots of two types of equations by means of addition-subtraction logarithms.[2] In the first part equations involving powers of x will be considered, while the second part will deal with exponential equations.

Formulas similar to those of the first part have been stated by Lüroth in the Rendiconti del Circolo Matematico di Palermo, vol. 27 (1909), pp. 393–401. Lüroth's formulas, however, are not adapted to simple tables such as those of Jones. As far as the writer knows, the formulas of the second part are new.

1. *Trinomial Equations.*

Let $gx^m+hx^n=k$ represent the equation under consideration. The method to be explained will give all of its real solutions provided each of the five constants is real. Any or all of the constants may be negative, fractional, or irrational. The only limitation is that there must be only two terms containing the unknown.

Case 1. $gx^m+hx^n=k$, *and* g, h, k, *and* x *are all positive.* The value of x to be found should be roughly estimated. The equation should be arranged so that, for this value of x, $gx^m>hx^n$. In deriving the formulas the terms gx^m and hx^n are considered the numbers to be added and k their sum. Then from the definition of the quantities A and B, as used in the table of addition-subtraction logarithms,

$$A = \log h - \log g - (m - n) \log x,$$

$$B = \log k - \log g - m \log x.$$

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[1] George W. Jones, *Logarithmic Tables*, The Macmillan Co. (Tenth Edition, 1905).

[2] *Editor's Note*: For an account of Gauss's method of solving certain algebraic equations by means of addition-subtraction logarithms, see his *Beiträge zur Theorie der algebraische Gleichungen* (Zweite Abteilung) in his Werke, vol. 3, p. 85; see also the paragraph entitled *Die Gauss'sche Methode der Auflösung der trinomischer Gleichungen* in Weber's *Lehrbuch der Algebra* (Zweite Auflage, 1898), pp. 393–397.

Elimination of $\log x$ between these two equations gives

$$(1) \qquad A - m^{-1}(m-n)B = \log h - \log g - m^{-1}(m-n)(\log k - \log g).$$

Solving the first of them for $\log x$ gives

$$(2) \qquad \log x = (\log h - \log g - A)/(m-n).$$

The value of each of the five given constants should be substituted in formula (1) and the formula simplified. A and B will be unknown. We find from the table of addition-subtraction logarithms the value of A which, with its corresponding value of B, satisfies formula (1). After A has been thus determined, we find $\log x$ from (2), and then find x from its logarithm.

Illustrative Example. Solve the equation $x^4+31.8x=28.72$. We shall proceed to find the positive root whose value is approximately .9. Since for this root $31.8x>x^4$, we arrange the equation in the form, $31.8x+x^4=28.72$. Then $g=31.8$, $h=1$, $m=1$, $n=4$, and $k=28.72$. From (1),

$$A-1^{-1}(1-4)\,B=\log 1-\log 31.8-1^{-1}(1-4)\,(\log 28.72-\log 31.8)$$

or

$$A + 3B = 0 - 1.502427 + 3(1.458184 - 1.502427).$$

This gives $A+3B=8.364844-10$. From the table of addition-subtraction logarithms we find that if $A=8.336-10$,

$$B = .009314, \text{ and } A + 3B = 8.363942 - 10;$$

if and $A=8.337-10$,

$$B = .009335, \text{ and } A + 3B = 8.365005 - 10.$$

Since $A+3B$ is supposed to equal $8.364844-10$, we find by interpolating that[3] $A=8.33685-10$. From (2) we get

$$\log x = (\log 1 - \log 31.8 + 10 - 8.33685)/(-3) = 9.94643 - 10.$$

Therefore $x=.88395$.

Case 2. $gx^m-hx^n=k$ and g, h, k, and x are all positive. The value of x to be found should first be approximated and a rough calculation should be made to see whether, for this value of x, $\log gx^m-\log hx^n$ is less than or greater than .4.

[3] Finding A to 5 decimal places from three-place tabulated values is justifiable, because the function $A+kB$, for any k, is almost a linear function of A for nearby values. The successive differences are large enough to determine two interpolated figures correctly—often 3. Thus, when

$A=8.335$, $A+3B=8.362876$
$\rightarrow$Diff. $=1066$
$A=8.336$, $A+3B=8.363942$
$\rightarrow$Diff. $=1063$
$A=8.337$, $A+3B=8.365005$
$\rightarrow$Diff. $=1063$
$A=8.338$, $A+3B=8.366068$

Case 2a. When $\log gx^m - \log hx^n < .4$. By making use of the definitions of A and B logarithms it may be shown as in case 1 that

$$A + n(m-n)^{-1}B = \log k - \log h + n(m-n)^{-1}(\log g - \log h), \tag{3}$$

and

$$\log x = (\log k - \log h - A)/n. \tag{4}$$

We find from the table of addition-subtraction logarithms the value of A which, with its corresponding value of B, satisfies formula (3). We determine $\log x$ from (4), and then find x from its logarithm.

Illustrative Example: Solve the equation $x^3 - 7x^{\frac{1}{2}} = 2$. We shall find the root between 2 and 3. For this root, $\log x^3 - \log 7x^{\frac{1}{2}} < .4$, so the equation comes under *Case 2a*. Here $g = 1$, $h = 7$, $k = 2$, $m = 3$, and $n = \frac{1}{2}$. Substituting in (3),

$$A + \tfrac{1}{2}(3 - \tfrac{1}{2})^{-1} \cdot B = \log 2 - \log 7 + \tfrac{1}{2}(3 - \tfrac{1}{2})^{-1}(\log 1 - \log 7),$$

or

$$A + \tfrac{1}{5}B = .301030 - .845098 + \tfrac{1}{5}(0 - .845098).$$

This gives $A + \frac{1}{5}B = 9.286912 - 10$. From the tables find a value of A which, when one fifth of the corresponding value of B is added, gives $9.286912 - 10$. We find $A = 9.27202 - 10$. From (4),

$$\log x = (\log 2 - \log 7 + 10 - 9.27202)/\tfrac{1}{2} = .36782.$$

Hence $x = 2.3325$.

Case 2b. When $\log gx^m - \log hx^n > .4$. For this case the B and C logarithms must be used. The formulas, found as in case 1, are:

$$B + m^{-1}(m-n) \cdot C = \log g - \log h + m^{-1}(m-n)(\log k - \log g), \tag{5}$$

and

$$\log x = (B - \log g + \log h)/(m-n). \tag{6}$$

We find from the table of addition-subtraction logarithms the value of B which, with its corresponding value of C, satisfies formula (5). We then find $\log x$ from (6), and x from its logarithm.

Illustrative Example: Solve the equation $(\sqrt{845})x^\pi - 93.26x^{-\sqrt{17}} = 1000$. This equation has purposely been made difficult to show the general utility of the method. There is a root between 3 and 4 which will be determined. Since for this root $\log [(\sqrt{845})x^\pi] - \log [93.26x^{-\sqrt{17}}] > .4$, the equation comes under *Case 2b*. Here $g = (\sqrt{845})$, $h = 93.26$, $k = 1000$, $m = \pi$, and $n = -\sqrt{17}$. From (5),

$$B + \pi^{-1}(\pi + \sqrt{17})C = \log \sqrt{845} - \log 93.26 + \pi^{-1}(\pi + \sqrt{17})(\log 1000 - \log \sqrt{845})$$

which gives $B + 2.3124C = 3.046902$. We locate a value of B so that the sum of

this B and 2.3124 times its corresponding C is 3.046902. That value of B is 3.0478. From (6) we get

$$\log x = (3.0478 - 1.463428 + 1.969695)/7.2647 = .4892.$$

Hence $x = 3.085$.

If a root is to be found under *Case* 1, it may save time in determining A to change formula (2) into

$$A = \log h - \log g - (m - n) \log x.$$

By substituting in this formula a rough approximation to x it is possible to get an idea as to just about what value of A to consider. Formulas of similar usefulness may be obtained for *Case* 2*a* by changing (4) into

$$A = \log k - \log h - n \log x,$$

and for *Case* 2*b* by changing (6) into

$$B = \log g - \log h + (m - n) \log x.$$

These supplementary formulas are especially helpful when it is desired to find two roots so close together that they both come under the same case. Thus, $7x - x^2 = 12$ comes under *Case* 2*a* for both $x = 3$ and $x = 4$. Using the formula suggested in this paragraph for *Case* 2*a*, we find that for the first root $A = .12494$ and for the second root $A = 9.87596 - 10$. These values both satisfy formula (3), and when substituted in formula (4) give, respectively, $x = 3.000$ and $x = 4.0000$.

Negative roots can be obtained, as is ordinarily done, by finding the corresponding positive roots of $f(-x)$. Hence x may be confined to positive values. If k is negative, the equation should be multiplied through by -1. Then either both the coefficients will be positive as in *Case* 1, or one will be positive and the other negative as in *Case* 2.

Limiting the number of terms containing powers of x to two does not greatly impair the usefulness of the method. Any cubic equation with integral exponents can be handled by making the proper substitution so as to eliminate the second degree term. Equations of higher degree often arise through the process of eliminating radicals or fractions. Since the method here proposed does not require exponents and coefficients to be integral, many of these higher degree equations can be avoided. Thus, $x^2 - x^{\frac{1}{2}} = 5$ would, upon changing to the form $x^2 - 5 = x^{\frac{1}{2}}$ and squaring each side, give a fourth degree equation with three terms containing x to which the logarithmic method of this paper does not apply. However, by leaving the equation in the form, $x^2 - x^{\frac{1}{2}} = 5$, the method applies directly.

The advantages of this logarithmic method over the more usual methods of solving equations containing powers of x may be summarized as follows: (1) It eliminates much of the tedious arithmetical work, by substituting for it the use of tables. (2) It is applicable to a large variety of such equations which

are otherwise solvable only by the process of successive guesses. (3) Answers can be found to 4 or 5 significant figures in much less time than by other methods. The writer's experience shows that it requires on the average about one half as much time as Horner's method.

2. *Exponential Equations.*

The logarithmic method is applicable also to exponential equations. Here we shall confine ourselves to only one special form, which occurs however in many different connections. It is

$$ge^{cx}+he^{-cx}=k,$$

where g, h, c, and k are any positive real numbers, $e=2.71828\cdots$, and $ge^{cx}>he^{-cx}$ for the value of x to be found. For this equation the following two formulas may be derived in much the same way as before:

$$A - 2B = \log g + \log h - 2\log k, \tag{7}$$

and

$$X = (\log h - \log g - A)/.86859c \tag{8}$$

The values of A and B are found by reference to the table of addition-subtraction logarithms. We determine the value of A which, with its corresponding value of B, satisfies formula (7). We find x by substituting this value for A in (8).

Two illustrative problems dealing with exponential functions will be solved.

Suppose it is desired to find the angle whose hyperbolic cosine is 3/2. Since $\cosh x=\frac{1}{2}(e^x+e^{-x})$, we have

$$.5e^x + .5e^{-x} = 1.5.$$

Here $g=.5$, $h=.5$, $c=1$, and $k=1.5$. From (7),

$$A - 2B = 2\log .5 - 2\log 1.5 = 9.045757 - 10.$$

From the table of addition-subtraction logarithms we find that if $A=9.164-10$, $B=.059140$, and $A-2B=9.045720-10$. If $A=9.165-10$, $B=.059267$, and $A-2B=9.046466-10$. Since $A-2B$ is supposed to be $9.045757-10$, we find by interpolating that $A=9.16405-10$. From (8), $x=-A/.86859=.96242$.

The equation commonly given for the catenary is $y=\frac{1}{2}a(e^{x/a}+e^{-x/a})$. Suppose we know that $a=60$, and want to determine x so that $y=183.7$. Here we have

$$30e^{x/60} + 30e^{-x/60} = 183.7,$$

for which $g=30$, $h=30$, $c=1/60$, and $k=183.7$. From (7),

$$A - 2B = 2\log 30 - 2\log 183.7 = 8.426025 - 10.$$

The value of A which, when 2 times the value of its corresponding B is subtracted from it, gives $8.426025-10$ is $A=8.45018-10$. From (8),

$$x = -A/.86859c = 1.54982/.014476 = 107.06.$$

Conclusion

No attempt has been made to give an exhaustive survey of the logarithmic method which has been explained. Enough has been included to show the general utility of the method in the solution of equations of many different types by the use of the same set of tables and the same procedure. Only one type of exponential equations has been solved, but the method can be readily extended to others.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON A PAPER CONCERNING A BESSEL FUNCTION

By S. A. SCHELKUNOFF, Bell Telephone Laboratories, Inc.

In a recent issue of this Monthly, Mr. C. C. Furnas presented an empirical formula for the modified Bessel function.[1] The method by means of which he arrived at this approximation is an interesting one, but his final formula and table are not so accurate as he believes them to be, partly because of the inherent limitations of the method itself and partly because of inaccuracies of mechanical integration. The purpose of this note is to call attention to a more accurate formula, and to correct the table for the benefit of those who are interested in practical applications of Bessel functions. Our method is less ingenious but more straightforward than his.

While the method used by Mr. Furnas leads to a series defining $\log I_0$, it appears that his interest is really centered in the values of $I_0(x)$ itself, since he tabulates the values of this latter function. If this is the case it would have been a fairly simple matter to make use of the well-known asymptotic expansion

$$(1) \qquad I_0(x) = \frac{e^x}{\sqrt{(2\pi x)}}\left[1+\frac{1}{8x}+\frac{9}{128x^2}+\frac{75}{1024x^3}+\cdots\right],$$

which may be found, for example, in Watson's *Theory of Bessel's Functions*, p. 203, equation (2).

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[1] C. C. Furnas, *Evaluation of the modified Bessel function of the first kind and zeroth order*, this Monthly, vol. 37 (1930), pp. 282–287.

If, on the other hand, there is some special reason for desiring a series for $\log I_0(x)$, we can easily obtain one by the simple expedient of taking the logarithm of (1). Thus we are led to

$$(2)\quad \log I_0(x) = x - \frac{1}{2}\log x - \frac{1}{2}\log 2\pi + \log\left[1 + \frac{1}{8x} + \frac{9}{128x^2} + \frac{75}{1024x^3} + \cdots\right]$$

or, expanding the last term in a series by means of the usual formula for $\log(1+\epsilon)$ when ϵ is small, to

$$(3)\quad \log I_0(x) = x - \frac{1}{2}\log x - \frac{1}{2}\log 2\pi + \frac{1}{8x} + \frac{1}{16x^2} + \frac{100}{1536x^3} + \cdots.$$

This process could readily be justified by introducing the remainder term in (2) and treating the resulting expression in the customary way.

Of course the series for $\log_{10} I_0$ may be obtained immediately by multiplying (3) throughout by $\log_{10} e$. For the purpose of comparison with Mr. Furnas' equation (16), it may be interesting to write down the first few terms of this series. They are:

$$(4)\quad \log_{10} I_0(x) = 0.43429\left[x + \frac{1}{8x} + \cdots\right] - \frac{1}{2}\log_{10} x - 0.39909.$$

The accompanying table was computed by the use of formula (1) and its accuracy checked by using (3) independently as a control. It is believed to be accurate to within half a unit in the last place.

Comparing our table with that given by Mr. Furnas, we observe that the differences between his values for $I_0(x)$ and ours range between -0.149 and $+0.111$ times the corresponding powers of 10. Considering the inherent limitations of the method used by Mr. Furnas it is not at all surprising that such inaccuracies should have crept in.

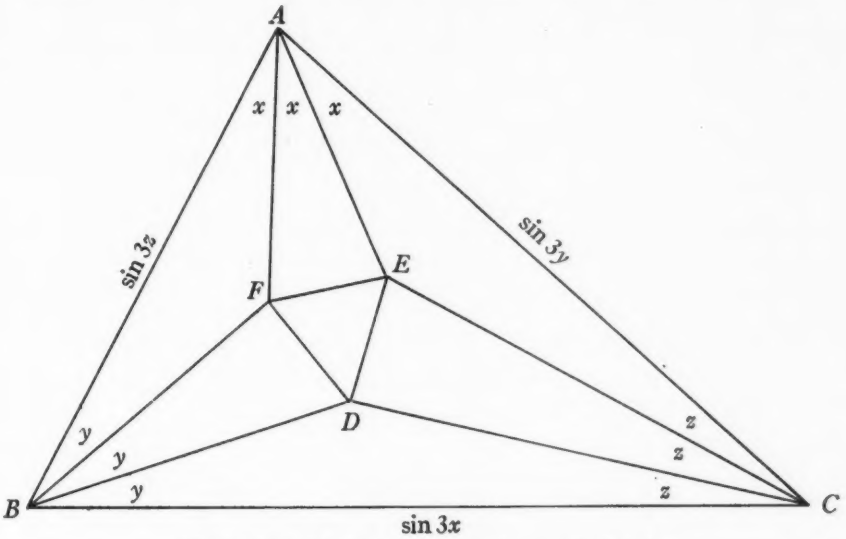
Values of the Modified Bessel Function $I_0(x)$

x	$I_0(x)$	x	$I_0(x)$
12	1.8949×10^{4}	90	5.1392×10^{37}
14	1.2942×10^{5}	100	1.0738×10^{42}
16	8.9345×10^{5}	200	2.0397×10^{85}
18	6.2184×10^{6}	300	4.4758×10^{128}
20	4.3558×10^{7}	400	1.0419×10^{172}
30	7.8167×10^{11}	500	2.5048×10^{215}
40	1.4895×10^{16}	600	6.1463×10^{258}
50	2.9326×10^{20}	700	1.5296×10^{302}
60	5.8941×10^{24}	800	3.8461×10^{345}
70	1.2016×10^{29}	900	9.7473×10^{388}
80	2.4752×10^{33}	1000	2.4857×10^{432}

A Simple Proof of the Theorem of Morley

By Jacob O. Engelhardt, Brooklyn, N. Y.

Theorem[1]: *If the three angles of a triangle be trisected, the triangle whose vertices are each the intersection of a pair of trisectors adjacent to a side, is equilateral.*



$2R=1$, where R is the radius of the circumscribed circle.

Lemma 1: $\sin 3a = 4\sin a \sin(60°+a)\sin(60°-a)$.

Lemma 2: If $a+b+c=180°$, $\cos^2 a+\cos^2 b+\cos^2 c = 1-2\cos(a)\cos(b)\cos(c)$.

Lemma 3: $x+y+z=60°$.

Proof:

$$BD = \frac{\sin 3x \sin z}{\sin (y+z)} = 4 \sin x \sin z \sin (60+x).$$

Similarly $BF = 4 \sin x \sin z \sin (60+z)$.

Therefore

$$(FD)^2 = 16 \sin^2 x \sin^2 z[\sin^2 (60+x) + \sin^2 (60+z) - 2 \sin (60+x) \sin (60+z) \cos y] = 16 \sin^2 x \sin^2 z[\cos^2 (30-x) + \cos^2 (30-z) - 2 \cos (30-x) \cos (30-z) \cos y]$$

[1] For other proofs of this theorem see R. A. Johnson, *Modern Geometry*, p. 253; and Philip Franklin, in Contributions of the Mathematics Department of the Massachusetts Institute of Technology, Second Series, No. 117 (Nov., 1926), p. 57.

But, by Lemma 2,

$$\cos^2(30-x)+\cos^2(30-z) = 1-\cos^2(180-y)-2\cos(30-x)\cos(30-z)\cos(180-y)$$
$$= \sin^2(180-y)+2\cos(30-x)\cos(30-z)\cos y.$$

Therefore $(FD)^2=16\sin^2 x\sin^2 z\sin^2(180-y)$ and $FD=4\sin x\sin y\sin z$. Similarly $DE=4\sin x\sin y\sin z$, $EF=4\sin x\sin y\sin z$, and therefore $FD=DE=EF$.

Construction of A Table of Hyperbolic Sines and Cosines

By S. A. Corey, Des Moines, Iowa

In the May, 1930 issue of this Monthly, Professor J. P. Ballantine[1] has given a method of computing a table of sines and cosines, which by making a slight modification may also be used in computing a table of hyperbolic sines and cosines.

Employing the identities,

$$\begin{aligned}\sinh(n+1)x &= \sinh(n-1)x+2\cosh nx\sinh x,\\ \cosh(n+1)x &= \cosh(n-1)x+2\sinh nx\sinh x,\end{aligned} \tag{1}$$

and taking

$$2\sinh x = (.1)^k, \tag{2}$$

where k may for convenience be 1, 2, or 3 we get by substituting in (1):

$$\begin{aligned}\sinh(n+1)x &= \sinh(n-1)x+(.1)^k\cosh nx,\\ \cosh(n+1)x &= \cosh(n-1)x+(.1)^k\sinh nx.\end{aligned} \tag{3}$$

Taking this value of x and tabulating $\cosh nx$ for even values of n and $\sinh nx$ for odd values of n, we get by taking $k=1$:

$$\begin{aligned}1. &= \cosh 0x\\ .05 &= \sinh 1x \quad \text{(From Equation (2))}\\ 1.005 &= \cosh 2x = \cosh 0x + .1\sinh 1x\\ .1505 &= \sinh 3x = \sinh 1x + .1\cosh 2x\\ 1.02005 &= \cosh 4x = \cosh 2x + .1\sinh 3x\\ .25250 &= \sinh 5x = \sinh 3x + .1\cosh 4x,\end{aligned}$$

and so on.

Each new entry is obtained by adding one-tenth of the last entry obtained to the entry next preceding. By taking k larger, our unit of measurement, $2x$, may be made as small as desired. As the unit of measurement decreases the tabular accuracy increases. Professor Ballantine's equation (4) would indicate that 20-place accuracy could be obtained by taking $k=7$.

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[1] Vol. 37 (1930), pp. 248–250.

An Elementary Upper Bound to the Roots of Equations

By J. M. Feld, Columbia University

1. *Introduction.* Cauchy has given a rule for the determination of an upper bound to the roots of algebraic equations, viz., the positive roots of

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

cannot exceed the greatest of the quantities

$$(ka_r)^{1/r},\ (ka_s)^{1/s},\ \cdots,\ (ka_t)^{1/t},$$

where $a_n, a_s \cdots, a_t$ are the negative coefficients and k is the number of negative coefficients. The purpose of this note is to extend this result to equations

$$\sum_0^m a_i e^{\alpha_i x} = 0.$$

Two cases will be considered. First an upper bound to the positive roots will be found, assuming the a's to be real; and secondly we shall determine an upper bound to the real part of the roots of exponential equations, allowing the a's to be complex. In both cases the α's are real.

2. Let

$$e^{\alpha_n x} + a_{n-1} e^{\alpha_{n-1} x} + \cdots + a_0 e^{\alpha_0 x} = 0, \tag{1}$$

where the α's and a's are real numbers and let $\alpha_n > \alpha_{n-1} > \cdots > \alpha$. In order that (1) have a positive root at least one of the a's must be negative. Transposing the terms with negative a's to the right we get

$$e^{\alpha_n x} + a_p e^{\alpha_p x} + \cdots + a_{p+i} e^{\alpha_{p+i} x} = a_q e^{\alpha_q x} + \cdots + a_{q+k-1} e^{\alpha_{q+k-1} x} \tag{2}$$

Since

$$e^{\alpha_n x}/k \geqq a_{q+j} e^{\alpha_{q+j} x}$$

if

$$x \geqq \frac{\log (ka_{q+j})}{\alpha_n - \alpha_{q+j}},$$

we see that

$$e^{\alpha_n x} \geqq a_q e^{\alpha_q x} + \cdots + a_{q+k-1} e^{\alpha_{q+k-1} x}$$

provided x is greater than or equal to the greatest of the quantities

$$\frac{\log (ka_{q+j})}{\alpha_n - \alpha_{q+j}} \qquad (j = 0, 1, \cdots, k-1). \tag{3}$$

Therefore (1) can have no positive root greater than the greatest of the quantities (3).

If the α's in (1) are integers such that $\alpha_k = k$, letting $e^{kx} = y^k$ (1) becomes

$$y^n + a_{n-1} y^{n-1} + \cdots + a_0 = 0; \tag{4}$$

and we have the result that (4) can have no positive root greater than the greatest of the quantities,

$$(ka_{q+j})^{1/(n-q-j)} \qquad (j = 0, 1, \cdots, k-1). \tag{5}$$

In the equation

$$e^{\alpha_n x} - ce^{\alpha_{n-1} x} - \sum_{i=2}^{n} n^{\beta_i} c^{\gamma_i} e^{\alpha_{n-i} x} = 0, \tag{6}$$

where c is positive and

$$\beta_i = \frac{\alpha_{n-1} - \alpha_{n-i}}{\alpha_n - \alpha_{n-1}}, \qquad \gamma_i = \frac{\alpha_n - \alpha_{n-i}}{\alpha_n - \alpha_{n-1}}$$

the quantities (3) all equal

$$\frac{\log (nc)}{\alpha_n - \alpha_{n-1}},$$

which is a root of (6).

3. Let the a's in (1) be complex. Then for any x that satisfies (1) we have

$$| e^{\alpha_n x} | \leqq | a_{n-1} e^{\alpha_{n-1} x} | + \cdots + | a_0 e^{\alpha_0 x} | . \tag{7}$$

If k is the number of terms in the right member of (7) we have

$$| e^{\alpha_n x} | > | a_{n-1} e^{\alpha_{n-1} x} | + \cdots + | a_0 e^{\alpha_0 x} |$$

for any x such that

$$| e^{\alpha_n x} | > k | a_{n-p} e^{\alpha_{n-p} x} | \qquad (p = 1, 2, \cdots, n); \tag{8}$$

and consequently such values of x cannot satisfy (1). From (8) we obtain

$$| e^{(\alpha_n - \alpha_{n-p}) x} | > k | a_{n-p} |$$

or

$$R(x) > \frac{\log k | a_{n-p} |}{\alpha_n - \alpha_{n-p}} \qquad (p = 1, 2, \cdots, n), \tag{9}$$

where $R(x)$ represents the real part of x. Thus (1) can have no roots in the half plane

$$R(x) > \text{the greatest } \frac{\log k | a_{n-p} |}{\alpha_n - \alpha_{n-p}} \qquad (p = 1, 2, \cdots, n).$$

If in (4) the a's are complex this result becomes: the absolute value of the roots of (4) cannot exceed the greatest of the numbers,

$$(k | a_{n-p} |)^{1/p} \qquad (p = 1, 2, \cdots, n).$$

Two Discontinuous Functions

By Marie M. Johnson, Oberlin College

1. A rectangular piece of wood is turned horizontally from one corridor into another of the same width and at right angles to the first. Consider pieces with one dimension constant and equal to a. What must be the width w of the corridors as the second dimension x of the wood varies? It is found that the function is defined analytically as follows:

$$w = 0, \qquad x = 0;$$

$$w = \left(x + \frac{a}{2}\right)\frac{\sqrt{2}}{2}, \qquad 0 < x \leqq a;$$

$$w = \left(\frac{x}{2} + a\right)\frac{\sqrt{2}}{2}, \qquad x \geqq a.$$

This function has a discontinuity at $x=0$, and the first derivative has discontinuities at $x=0$ and $x=a$.

2. Let the first problem be modified in the following manner: The piece of wood is turned around the corner of the corridors only when this permits the width of the corridors to be less than when the piece is not turned. This function is represented by the formula

$$w = 0, \qquad x = 0;$$

$$w = \left(x + \frac{a}{2}\right)\frac{\sqrt{2}}{2}, \qquad 0 < x \leqq \frac{2\sqrt{2}-1}{2}a;$$

$$w = a, \qquad \frac{2\sqrt{2}-1}{2}a \leqq x \leqq a;$$

$$w = x, \qquad a \leqq x \leqq \frac{2}{2\sqrt{2}-1}a;$$

$$w = \left(\frac{x}{2} + a\right)\frac{\sqrt{2}}{2}, \qquad x \geqq \frac{2}{2\sqrt{2}-1}a;$$

and it is discontinuous at $x=0$. The first derivative has a number of discontinuities.

A Note by the Editor

Another excellent example of a discontinuous function will be found in an article in *The Mathematics Teacher* for December, 1927, entitled, *Functions in General, and the Function* [x] *in Particular*, by Professor Walter B. Carver of Cornell University.

On Collateral Reading in Mathematics

By J. H. Kusner, University of Florida

In the May, 1928 issue of this Monthly[1] there appeared a report of a committee of the Mathematical Association of America, presenting a list of books for use as collateral reading by freshmen and sophomores, together with a list of assignments and a proposed method of administration. These suggestions struck a responsive chord in the mind of the present writer and he has for the past two years been experimenting with collateral reading, and has, he believes, learned one or two things worthy of communication.

Probably the main objection which teachers of college mathematics have to such innovations as collateral reading is the combined one of the difficulty of working the reading in as an integral part of the course and also the objection to taking time away from the study *of* mathematics for the purpose of studying *about* mathematics and mathematicians. The committee proposed that the instructor devote a small portion of time, say one half hour per week, to class discussion of the material of the reading, which not only has the objection of taking class time, but also necessitates a multiplicity of copies of reserve books since all members of the class must read the same assignment each week. Also, it involves making the collateral reading required work, which many teachers who are in sympathy with the general idea are not ready to do.

The present writer has developed a method of administration which, although undoubtedly capable of much improvement, has little in it that is objectionable on the grounds mentioned above.

First of all, the reading has been placed on a voluntary basis. At the beginning of each semester, the student is offered the opportunity of entering upon the course of reading *in addition* to the regular work of the course and entirely separate from it, and if he elects it he must read one assignment each week. Naturally, so long as there exists the necessity of assigning grades, it is only just that the satisfactory completion of the collateral reading have some effect on the grade awarded. The present writer has adopted the scheme of raising the semester grade one letter if the reading for the entire semester has been completed satisfactorily and if the student has already earned a passing grade in the course. Thus a student earning D, C, B, or A in the regular work of the course receives C, B, A, or A respectively if he does the reading. In this way, no student receives credit for the regular course as a result of collateral reading alone, and yet some recognition is awarded, except for the A student, who generally is motivated strongly enough by intellectual curiosity to need no recognition.

The weekly discussion in class suggested by the committee has been replaced by a five to ten minute discussion between the student and instructor on the completion of an assignment, these discussions taking place during a regular

[1] Vol. 35 (1928), pp. 221–228.

office hour, the student appearing whenever he has a report to make. Many teachers will, of course, object to the great burden of so many personal conferences, and the writer can do little but agree that it is a great burden if the instructor has very many students doing the reading. Capacities, however, vary widely, and if the instructor is not overburdened he will find these collateral reading conferences very interesting and enjoyable. Indeed, the present writer was most pleasantly surprised when he first undertook this work and found that even the duller students have their interest and appreciation awakened as a result of the readings and conferences. In fact he has come to believe that some of his best work has been done by means of the close and effective personal contact which this makes possible. Rare indeed are the students in whom there is no response to the compelling appeal of the story of mathematics and its creators to be found in the course of collateral reading, and the opportunity for vital guidance on the part of the instructor in the report conference is greater than it seems to be before it has been experienced. The present writer feels that the report conference is perhaps the most valuable element of the collateral reading course, and he recommends it strongly.

Further, the assignments themselves, as indicated in the appended copy of the latest edition of the list used by the writer, have been modified somewhat, and for two reasons. First, there is too much duplication in the books of the original list. Second, it has been found to be more satisfactory to have in the first few assignments a survey of the history of mathematical science so that the student may soon receive a perspective picture into which may fit the more detailed historical and biographical readings which are placed later in the list. This seems to develop an earlier hold on the interest of the all too prevalent type of student who feels—"what good is mathematics anyhow?"

It is only these first few assignments which need be read in a definite order. The later historical readings fit into the picture already obtained and the philosophical and miscellaneous readings are satisfactory anywhere. There is the additional advantage of this plan that fewer reserve copies of the books are needed. The writer recommends one copy per six students of the books of assignments 1 to 6, one copy per twelve students of the books of assignments $A7$ and $B7$, and one copy per seventy-five students of each of the others.

It is obvious that a number of books suitable for this purpose are missing from the list. This is due solely to library limitations and delays involved in securing out-of-print books. The list will, of course, grow as time passes inasmuch as books satisfactory for this purpose are appearing with ever increasing frequency.

In conclusion, it should be remarked that some of the reading list is usable even among secondary school students. Some years ago the present writer experimented with collateral reading with boys and girls fourteen and fifteen years old, and found that if done judiciously, good results could be obtained.

The following is a copy of the reading list which has been issued to students taking this work.

University of Florida—Collateral Readings in Mathematics

Mathematics is, par excellence, the science of exact thought. As such, it is the natural study of all persons possessing the attribute of intellectual curiosity, those seeking entrance to the life of the mind. Furthermore, as the language of the deductive sciences, its role in modern life is a very important one.

However, the place of mathematics in a liberal education rests not only upon these considerations, but also upon those of another sort. In the words of a great philosopher and mathematician, "Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry." (Bertrand Russell in "Mysticism and Logic.")

Unfortunately, in most cases this exaltation comes but gradually—and even then, generally only after rather extensive study. However, a *thoughtful and critical* consideration of certain aspects of mathematics and mathematicians does make it possible to obtain more quickly a glimpse of what Wordsworth describes as "that independent world created out of pure intelligence." It is these aspects which are considered in the readings listed below.

One assignment is to be read each week and a five or ten minute oral discussion with the instructor will be necessary upon the completion of each assignment. All the books listed are on reserve in the main reading room of the Library.

The first ten assignments must be read in the order in which they appear in the list. The remaining ones may be read in any order after the completion of the first ten.

1. *History of Elementary Mathematics*—F. Cajori. Pp. 1–92.
2. *Mathematics* (*Our debt to Greece and Rome*)—D. E. Smith. Pp. 1–89 and 160–164.
3. *History of Elementary Mathematics*—F. Cajori. Pp. 93–138.
4. *History of Mathematics in Europe*—J. W. N. Sullivan. All.
5. *Early Mathematics in North and South America*—F. Cajori. Pp. 11–86.
6. *Early Mathematics in North and South America*—F. Cajori. Pp. 90–149.

There are two alternatives for 7, 8, 9, 10.

A. 7. *Pioneers of Science*—Sir Oliver Lodge. Pp. 5–107.
   8. *Pioneers of Science*—Sir Oliver Lodge. Pp. 108–202.
   9. *Pioneers of Science*—Sir Oliver Lodge. Pp. 206–303.
   10. *Pioneers of Science*—Sir Oliver Lodge. Pp. 305–397.

B. 7. *Makers of Science*—I. B. Hart. Pp. 19–102.
   8. *Makers of Science*—I. B. Hart. Pp. 103–209.
   9. *Makers of Science*—I. B. Hart. Pp. 210–314.

10. Read either 11 or 12 or 13 below.

These may be read in any order:

11. *Galileo*—W. W. Bryant. All.
12. *Kepler*—W. W. Bryant. All.
13. *Archimedes*—Sir T. L. Heath. All.
14. *Introduction to Mathematics*—A. N. Whitehead. Pp. 7–41.
15. *Introduction to Mathematics*—A. N. Whitehead Pp. 42–86.
16. *Philosophy of Mathematics*—J. B. Shaw. Pp. 1–30.
17. *Philosophy of Mathematics*—J. B. Shaw. Pp. 154–168 and 186–195.
18. *Science and the Modern World*—A. N. Whitehead. Pp. 1–54.
19. *Thinking about Thinking*—C. J. Keyser. All.
20. *The Fourth Dimension Simply Explained*—H. P. Manning. Pp. 7–51.
21. *Mysticism and Logic*—B. Russell. Pp. 58–73 and 33–45.
22. *The Human Worth of Rigorous Thinking*—C. J. Keyesr. Pp. 1–60.
23. *The Human Worth of Rigorous Thinking*—C. J. Keyser Pp. 271–314
24. *Mathematical Recreations and Essays*—W. W. R. Ball. Pp. 44–61 and 170–187.
25. *Canterbury Puzzles*—A. E. Dudeney. Pp. 11–57.
26. *Canterbury Puzzles*—A. E. Dudeney. Pp. 58–109.
27. *Canterbury Puzzles*—A. E. Dudeney. Pp. 110–161.
28. *The Beautiful Necessity*—Claude Bragdon. All.
29. *Projective Ornament*—Claude Bragdon. All.
30. *The Curves of Life*—T. A. Cook. Pp. 1–40.
31. *The Curves of Life*—T. A. Cook. Pp. 41–80.
32. *The Curves of Life*—T. A. Cook. Pp. 81–114.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y. and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Number, the Language of Science. A Critical Survey written for the Cultured Non-Mathematician. By Tobias Dantzig. New York, The Macmillan Company, 1930. xiv+260 pages. $3.50.

La Gamme. Introduction a l'Etude de la Musique. By P. J. Richard. Preface by Marius Casadesus. Paris, Librairie Scientifique Hermann et Cie, 1930. viii+232 pages. Paper, 28 francs.

Applied Mechanics. By Norman C. Riggs. New York, The Macmillan Company, 1930. xii+328 pages. $3.75.

Mechanics for Students of Physics and Engineering. By Henry Crew and K. K. Smith. New York, The Macmillan Company, 1930. xviii+372 pages. $4.00.

Intermediate Mechanics: Dynamics. By D. Humphrey. London, Longmans Green and Co., 1930. xii+382 pages. $4.20.

A Course of Analysis. By E. G. Phillips. Cambridge, The University Press, 1930. viii+362 pages.

Cours d'Analyse, Professé a l'École Polytechnique. Par J. Hadamard. Tome II. Paris, Librairie Scientifique Hermann et Cie, 1930. vi+722 pages. Paper, 140 francs.

The Principles of Quantum Mechanics. By P. A. M. Dirac. Oxford, The Clarendon Press, 1930. x+258 pages. $6.00.

Infinite Series. By Tomlinson Fort. Oxford, The Clarendon Press, 1930. iv+254 pages.

Mathematical Introduction to Economics. By Griffith C. Evans. McGraw-Hill Book Company, Inc., 1930. New York, xii+178 pages. $3.00.

Eléments de Trigonométrie Sphérique. Par G. Papelier. Paris, Librairie Vuibert, 1930. 166 pages. Paper, 20 francs.

Vorlesungen über Grundlagen der Geometrie. Von Kurt Reidemeister. Berlin, Julius Springer, 1930. Grundlehren der mathematischen Wissenschaften, Band XXXII. x+148 pages. Paper 11 marks, bound 12.60 marks.

Curve Piane Speciali Algebriche e Trascendenti. Teoria e Storia. By Gino Loria. Vol. II.—Curve Trascendenti, Curve Dedotte da Altre. Primo Edizione Italiana. Milan, Ulrico Hoepli, 1930. xii+440 pages.

Darstellende Geometrie. I: Elemente, Ebenflächige Gebilde. By Robert Haussner. Sammlung Goschen, 142. Berlin, Walter de Gruyter & Co., 1929. 208 pages.

Fouriersche Reihen. By Werner Rogosinski. Sammlung Goschen, 1022. Berlin, Walter de Gruyter & Co., 1930. 136 pages.

Elementary Theory of Finite Groups. By L. C. Mathewson, under the editorship of J. W. Young. Boston, Houghton Mifflin Co., 1930. x+166 pages. $2.50.

Differentialgleichungen Reeller Funktionen. By E. Kamke. Leipzig, Akademische Verlagsgesellschaft M.B.H., 1930. xiv+436 pages.

REVIEWS

Automorphic Functions. By Lester R. Ford. New York, McGraw-Hill Book Company, 1929. xii+333 pages. Price $4.50.

This book is the first containing an extensive systematic treatment of the theory of automorphic functions in English. The author has succeeded in presenting this difficult subject in a manner which makes it accessible to those who are familiar with the fundamentals of the theory of functions of a complex variable.

The first chapter, entitled "Linear Transformations," and the second, "Groups of Linear Transformations," give an especially clear treatment of topics which are of great importance in later work. The isometric circle, a concept introduced at an early stage, proves to be a valuable tool.

The third chapter is devoted to "Fuchsian Groups," and the fourth to some of the fundamental properties of automorphic functions.

In the fifth chapter existence theorems are established by means of the Poincaré theta series, and some properties of the theta functions are proved.

The sixth chapter, "The Elementary Groups," contains also a treatment of inversion in a sphere and of stereographic projection, as well as an interesting discussion of the regular solids.

Chapter VII,"The Elliptic Modular Functions," contains the definitions and some of the properties of the functions $J(\tau)$ and $\lambda(\tau)$ and a discussion of their relation to each other.

Chapter VIII gives a thorough treatment of conformal mapping, including the mapping of a plane simply connected region on a circle (with particular attention to the behavior of the mapping function on the boundary), the mapping of limit regions, and of simply connected finite-sheeted regions. These results are applied in Chapters IX and X to problems in uniformization.

The text closes with a brief discussion of the relation between the theory of automorphic functions and certain parts of the theory of linear homogeneous differential equations of the second order.

Not the least valuable part of the book is an excellent bibliography containing more than three hundred titles. The arrangement is chronological, but it is easy to locate all the references to a particular author, if desired, since the Author Index refers to the bibliography as well as to the text itself.

FRED W. PERKINS

Einführung in die Nicht-Euklidische Geometrie. By Hans Mohrmann. Akademische Verlagsgesellschaft M.B.H. Leipzig, 1930. XII+126 pages.

This introduction to non-Euclidean geometry by Professor Mohrmann makes very pleasant and interesting reading on account of its aggressiveness and boldness of point of view. In the discussions of the philosophical implications of non-Euclidean and its relation to Euclidean geometry one in almost reminded of the late Study's vigorous style.

The first three chapters are given in the main to such meditations and historic accounts. Here, as usual with many German authors, the importance of Gauss in the development of non-Euclidean geometry is stressed beyond reason. Lobatchewsky and Bolyai are so to speak merely mentioned "en passant." Again it seems to the reviewer that altogether too much credit is given to the philosopher Cornelius for his alleged remarkable evaluation of the essence of Euclidean geometry. Mohrmann dedicates the book to this Cornelius and maintains with the latter that Euclidean geometry is the only geometry of reality (Wirklichkeit). Anybody that cannot see that is mistaken! This matter has of course been controversial for a long time, but Cornelius' dogmatic position is by no means secure.

On the other hand the criticism on axiomatic research and its importance for geometric development is well deserved. As Study, Poincaré and others have pointed out on various occasions the over-emphasis of axiomatics since Hilbert's "Foundations" has done as much harm as good to geometric advancement.

Mohrmann makes some very sarcastic comments on the extreme "axiomatists" or "Mathematomanen."

The chapters which deal with Euclidean, projective and non-Euclidean geometry proper, are very well done. The position that analysis furnishes the only rigorous foundation for geometry is well taken.

It has moreover the advantage of great simplicity in the establishment of the groundwork of geometry. The reader will find Mohrmann's method of exposition of the foundations very direct and simple and for this reason very interesting.

The mapping process of the projective plane upon the Steiner-surface and by topological deformation upon the Steinitz-octatredron is new to the reviewer.

Various models of non-Euclidean geometries are discussed and the Cayley-Klein metrics is given adequate treatment. All this is done in a clear and competent manner.

Mohrmann's book should thus prove a welcome addition to the existing treatises on non-Euclidean geometry. It is written in a refreshing and fearless style which will stimulate independent thinking and create a critical attitude toward orthodox and dogmatic statements even when issued by acknowledged authorities.

ARNOLD EMCH

Leçons sur les Nombres Transfinis, By Waclaw Sierpinski; Gauthier-Villars, Paris, 1928. 240 pages.

This volume appears as a unit in the Borel series of monographs on function theory. In the preface, signed by Professor Borel himself, it is very aptly remarked that Professor Sierpinski requires no introduction to students of the theory of aggregates. His contributions to that theory, appearing with especial frequency in the *Fundamenta Mathematica*, are impressive. As is well known, the Polish school, which has developed the *Fundamenta* so successfully within a mere ten or twelve years, is primarily interested in problems of analysis situs admitting resolution by the methods of the theory of aggregates. Since Professor Sierpinski is one of the outstanding members of the school, it is a little surprising that he chose the subject of transfinite numbers for his monograph. An account, from his pen, of principles and methods in analysis situs *a la Polonaise* could not fail to afford exceptionally instructive reading. It would also provide an admirable companion volume for the Lefschetz monograph.

No scholar, however, is more competent than Professor Sierpinski to produce a really scientific treatise on transfinite numbers. The present work is precisely that. Moreover, the exposition is well arranged, fluent, and pleasing. The reviewer has no hesitation in commending this book to the attention of all who wish to acquaint themselves, from the most recent point of view, with fundamentals of the theory treated.

Part 1, pages 1–138, is particularly interesting. Here the full cognizance

taken of recent researches, published principally in the *Fundamenta*, affects even the point of view and the entire course of the exposition. Thus we find the important Bernstein theorem based upon a new theorem due to Banach (*Fundamenta*, vol. 6, page 236), for which a proof is given by Sierpinski. In fact, the discussion of inequalities between cardinal numbers becomes, in several respects, quite noteworthy. In this connection, for example, we find not only the König theorem, but also a modification, and partial extension, which leads to interesting conclusions. Numerous other features of the six chapters of Part 1 result directly from the advanced scientific outlook of the author. A conspicuous achievement is the adroit handling of the vexatious Zermelo principle of selection (Allgemeines Auswahlprinzip). Without committing himself in this controversial matter, Professor Sierpinski directs attention to many theorems, some of them very surprising indeed, which can not be established unless the Zermelo postulate is accepted. He also shows the equivalence of this postulate and the affirmative solution of the problem of trichotomy.

Following the analysis of cardinal numbers in Part 1, we find, in Part 2, a systematic and readable exposition of the theory of ordinal numbers. The author begins with a discussion of types of simple order, and of the Dedekind-process extension of dense to continuous types. He then proceeds quickly to well-ordered types and the associated ordinal numbers. Having studied these numbers along familiar lines, he presents the system of classes of ordinals and their corresponding cardinals (Alephs). The existence of a well-ordering for an arbitrary aggregate is exhibited as equivalent to the affirmative solution of the problem of trichotomy and to the Zermelo principle of selection. An excellent unsolved problem, due to Souslin (*Fundamenta*, vol. 1, page 223) is re-recorded on page 153.

The typography is good, and there are apparently no serious errors.

LESTER S. HILL

Grundlagen der Analysis. Das Rechnen mit ganzen, rationalen, irrationalen, komplexen Zahlen. By Edmund Landau. Leipzig, Akademische Gesellschaft, 1930, xiv+134 pages.

Mastering the theory of the real number system is one of the severest tests to which students of higher mathematics are put. It is not easy for the teacher to convince his class that he is serious when he defines a rational number as an ordered pair of integers, or a real number as a class of rational numbers. Uprooting the student's haphazard notions of number, and replacing them by the theories of Dedekind, Cantor and Weierstrass is a maneuver which has its risks. The prognosis is favorable only when the patient has high vitality at the outset.

Professor Landau's little book looks as if it may have the qualities of a specific. Certainly no clearer treatment of the foundations of the number system

can be offered to a student. Starting with Peano's five axioms for the natural numbers, Landau develops, with the utmost elegance, precision and completeness, the theories of the natural numbers, fractions, real numbers and complex numbers. Never before has this subject been treated with such explicitness.

The book is written in the crystalline style for which Landau is so well known. There are two prefaces, one for students, one for authorities. Speaking to the student, whom he "duzes," Landau says: "Bitte vergisz was du auf der Schule gelernt hast, denn du hast es nicht gelernt."

One can only be thankful to the author for this fundamental piece of exposition, which is alive with his vitality and genius.

J. F. RITT

Projective Geometry. By John Wesley Young. The Carus Mathematical Monographs, number four. The Open Court Publishing Company, 1930. ix+185 pages.

It may be stated at the outset that this fourth volume of the Carus Mathematical Monographs will prove to be an excellent auxiliary text for students in a first course of projective geometry.

The decision of the author "to develop on an intuitive basis the concepts and the properties of projective space, without any admixture of metric ideas," is very fortunate. This idea is carried out in five chapters in which topics like perspective drawing, projection and section, correspondence, projective transformations, the principle of duality, Desargues' theorem, the fundamental theorem, Pascal's and Brianchon's theorems, etc., are being treated.

In chapter VI metric properties considered from a projective standpoint are discussed: parallel lines, midpoint, the classification of conics, perpendicular lines, the orthogonal involution, angle bisectors, axes of conics, foci, construction of an involution by means of circles, metric definitions of a conic. This is always particularly welcomed by the beginning student, because it forms for him a bridge between the old and the new soil.

The next three chapters bear a more theoretical aspect, but are of the utmost importance for a more advanced understanding of projective geometry. Thus chapter VII deals with groups of projective transformations, chapter VIII with the algebra of points and the introduction of analytic methods, and the concluding chapter with groups and geometries.

That the group-concept is here brought to the foreground is a proof for the modern standpoint which the author is taking in the teaching of even elementary projective geometry. In this manner the fact that projective geometry and its analytic formulation may be established without any Euclidean metric concepts whatsoever may be made clear by a relatively simple method.

This is done in an excellent fashion. But it seems to the reviewer that the exposition of this important fact could be simplified still more by stressing the important role of analysis in this demonstration. I have in mind the mechanism of projective coordinates $(A_1A_3E_2P_2)=x=x_1/x_3$, etc., based upon von Staudt's

harmonic quadrangle − construction + projectivity + continuity axiom, in which the Cartesian plane is implicitly contained by making a special assumption about the infinite region of the plane. The complete analytic equivalence only assures absolute rigor.

The last chapter gives the student a bright outlook on the geometries which are associated with various groups. The groups of collineations which leave given conics invariant offer the simplest and shortest approach to non-Euclidean geometry. This fact alone suffices to show the importance of the group-concept in geometry. Thus, the author concludes his monograph with a very stimulating subject which will urge the good student to penetrate farther into the beautiful realms of geometry.

The book is very clearly written and typographically well done. The reviewer has noticed very few errors. In the historic account perhaps too much emphasis is placed upon Poncelet with the exclusion of some equally important pioneers in this field.

But these criticisms are not important enough to detract from the value of the very creditable monograph which Professor Young has written.

ARNOLD EMCH

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3456. *Proposed by C. A. Rupp, Pennsylvania State College.*

Show that the equation

$$\sum_k \frac{a_{ij}}{a_{ii}a_{jj}}x_k = 0, \quad (a_{ij} = a_{ji}, \ ijk \text{ a permutation of } 123)$$

represents a Steiner-Plücker line of the Pascal hexagons whose vertices are the intersections of the conic of equation

$$\sum a_{ij}x_ix_j = 0 \quad (i, j = 1, 2, 3)$$

with the sides of the coordinate triangle.

3457. *Proposed by E. T. Bell, California Institute of Technology.*

If $\binom{m}{r}$ denotes the coefficient of x^r in $(1+x)^m$ when m is an integer >0, and $\binom{0}{0}$ denotes 1, show that if n is an integer >0,

$$2\sum_{i=0}^{n} 2^{2n-2i}(-1)^i\binom{2j}{j}\binom{n}{j}\binom{n+j}{j} = [1+(-1)^n]\binom{n}{n/2}^2.$$

3458. *Proposed by J. Rosenbaum, Milford, Connecticut.*

In the triangle ABC, a tangent is drawn to the incircle cutting the lines BC and AC at X and Y respectively. Find the locus of the intersection of AX and BY.

3459. *Proposed by Norman Anning, University of Michigan.*

It is observed that $3003 = \binom{15}{5} = \binom{14}{6}$. Solve in positive integers the equation

$$\binom{x+1}{y} = \binom{x}{y+1}$$

3460. *Proposed by B. F. Kimball, University of New Hampshire.*

Show that the integral:

$$V_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^r + x_2^r + \cdots + x_n^r}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n$$

converges to the limit $2(r+1)^{-1}$ when n increases without limit, and that the product $n[V_n - 2(r+1)^{-1}]$ remains bounded; r to be any real number greater than -1. A generalization of 3408[1930, 38].

3461. *Proposed by Boyd C. Patterson, Hamilton College.*

Given a triangle $A_1A_2A_3$, its circumcircle (O), and a line L in its plane. To prove the following construction of the point P on (O) whose Simson line with respect to $A_1A_2A_3$ is parallel to L: Through the circumcenter O draw a line OX parallel to L. If M is one of the points where OX cuts (O) then P may be located by the relation

$$\text{arc } MP \equiv \text{arc } MA_1 + \text{arc } MA_2 + \text{arc } MA_3 \pmod{4\pi}$$

3462. *Proposed by R. C. Colwell and O. R. Ford, West Virginia University.*

A circular hoop, which is free to move on a smooth horizontal plane, has sliding on it a small ring of $1/n$th its mass, the coefficient of friction between the two being μ. Initially the hoop is at rest and the ring has an angular velocity ω round the hoop. Show that the ring comes to rest relative to the hoop after a time $(1+n)/\mu\omega$.

SOLUTIONS

3350. [1928, 494] *Proposed by Frank Irwin, University of California.*

Find the sum of all products of n factors each, the factors being chosen from the numbers 2, 3, 4, $\cdots$, $2+t$. Repetitions of factors are allowed; order is not considered; and two products $2\cdot 6$ and $3\cdot 4$ are regarded as distinct, though numerically equal.

Solution by Emma T. Lehmer, Brown University.

The problem is equivalent to finding the coefficient E_n^m of x^{m-n} in the polynomial

$$P(x, m) = (x + 2)(x + 3) \cdots (x + m + 1) = \sum_{k=0}^{m} E_{m-k}^m x^k, \quad E_0^m = 1,$$

where $t+1$ is replaced by m.
If we substitute $x+1$ for x in the above and multiply the result by $x+2$, we get

$$(x + 2)P(x + 1, m) = P(x, m + 1),$$

or

$$(x + 2) \sum_{k=0}^{m} E_{m-k}^m (x + 1)^k = \sum_{k=0}^{m+1} E_{m+1-k}^{m+1} x^k.$$

Equating coefficients on both sides of this last equation, we have

$$(1) \quad E_n^{m+1} = E_n^m + \sum_{k=m+1-n}^{m} \left[2\binom{k}{m+1-n} + \binom{k}{m-n} \right] E_{m-k}^m \quad (n = 1 \cdots m).$$

$$E_{m+1}^{m+1} = 2 \sum_{k=0}^{m} E_{m-k}^m.$$

Since $(x+m+2)P(x, m) = P(x, m+1)$, we find by equating coefficients, the following relation:

$$(2) \quad E_n^{m+1} = (m + 2)E_{n-1}^m + E_n^m \quad (n = 1 \cdots m)$$

$$E_{m+1}^{m+1} = (m + 2)E_m^m.$$

From (1) and (2) we derive

$$(3)\ 2\binom{m}{n} + \binom{m}{n+1} = -\sum_{k=1}^{n-1} \left[2\binom{m-k}{n-k} + \binom{m-k}{n-k+1} \right] E_k^m + nE_n^m \quad (n = 1 \cdots m).$$

We now solve by determinants the set of m equations (3), of which the following are the first three:

$$2\binom{m}{1} + \binom{m}{2} = E_1^m,$$

$$2\binom{m}{2} + \binom{m}{3} = -\left[2\binom{m-1}{1} + \binom{m-2}{2} \right] E_1^m + 2E_2^m,$$

$$2\binom{m}{3} + \binom{m}{4} = -\left[2\binom{m-1}{2} + \binom{m-1}{3} \right] E_1^m - \left[2\binom{m-2}{1} + \binom{m-2}{2} \right] E_2^m + 3E_3^m.$$

Hence

$$E_n^m = \frac{1}{n!}\begin{vmatrix} \left[2\binom{m}{1}+\binom{m}{2}\right] & -1 & 0 & \cdots & 0 \\ \left[2\binom{m}{2}+\binom{m}{3}\right]\left[2\binom{m-1}{1}+\binom{m-1}{2}\right] & -2 & & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ \left[2\binom{m}{n}+\binom{m}{n+1}\right]\left[2\binom{m-1}{n-1}+\binom{m-1}{n}\right]\left[2\binom{m-2}{n-2}+\binom{m-2}{n-1}\right] & \cdots & \left[2\binom{m-n+1}{1}+\binom{m-n+1}{2}\right] & & \end{vmatrix}$$

The first three of these functions of m are

$$E_1^m = \frac{1}{2}m(m+3), \quad E_2^m = \frac{m(m-1)(3m^2+17m+26)}{24},$$

$$E_3^m = \frac{m(m-1)(m-2)(m+3)(m^2+5m+8)}{48}.$$

3405 [1930, 37]. *Proposed by Paul Wernicke, Washington, D. C.*

Given in a plane three concurrent lines and a point P. Construct an equilateral triangle having its summits on the three lines and having the point P on one of its sides. Determine the number of solutions.

Solution by Lloyd C. Bagby, Linsly Institute of Technology.

Let l_1, l_2, l_3 be the three straight lines concurrent in O. On any one of the lines, say l_2, select a point Q, not at O, and construct the equilateral triangle QMN with the side MN lying upon l_1. Draw the straight lines NN', MM' so that they pass through Q', the symmetric of Q with respect to l_1. Then NN' and MM' are the loci of the third vertices of all equilateral triangles having one vertex at Q and another on l_1. Let NN' and MM' cut l_3 in N' and M', respectively. Then N' and M' determine two equilateral triangles with the sides QN' and QM', respectively, whose third vertices lie upon l_1. There are then three triangles homothetic to each of these triangles having a given point P on one of their sides or on a side extended. There are therefore six solutions.

Also solved by E. M. Berry, J. D. Leith, J. H. Neelley, A. Pelletier, J. A. Powers, O. J. Ramley, J. Rosenbaum, S. Kullback, and the Proposer.

3409 [1930, 94]. *Proposed by Norman Anning, University of Michigan.*

When $3b^2 > 8ac$, the curve, $y = ax^4+bx^3+cx^2+dx+e$, has two real inflectional tangents. Prove that they cut off equal areas.

Solution by Andrew G. Clark, Colorado Agricultural College.

If $F(x)$ is a polynomial, the area cut from $y = F(x)$ by the chord through the points of intersection of the two fixed lines, $x = x_1$ and $x = x_2$, with $y = F(x)$ is

invariant to the arbitrary character of the coefficients of the first and zero degree terms of $F(x)$.

This theorem holds providing the chord does not intersect $y=F(x)$ for $x_1<x<x_2$. The proof becomes evident when the expression,

$$\int_{x_1}^{x_2} F(x)dx - \tfrac{1}{2}(x_2 - x_1)[F(x_2) + F(x_1)],$$

for the area cut off by the chord is examined and found independent of the coefficients of the first and zero degree terms of $F(x)$.

The quartic to be considered loses nothing in generality if transformed into $y=f(x)=x^4+cx^2+dx+e$; it is clear that for the equation in this form, $c<0$ is the condition for two real points of inflection. Now, for convenience, replace c by $-6m^2$, whence

$$y = x^4 - 6m^2x^2 + dx + e;\ y' = 4x^3 - 12m^2x + d;\ \text{and}\ y'' = 12x^2 - 12m^2.$$

It is obvious, then, that the inflection points are $m, f(m)$ and $-m, f(-m)$, and that the equation of the tangent through the former is $y-f(m)=f'(m)\ (x-m)$. Now solving this equation with $y=f(x)$, we get the equation

$$f(x) - f(m) - f'(m)(x - m) = 0,\ \text{or}\ (x - m)^3(x + 3m) = 0.$$

The roots of this cubic are immediately seen to be independent of the coefficients d and e, in $f(x)$.

Therefore, if it can now be shown that for some particular values of d and e, the areas cut off by the two inflectional tangents are equal, an immediate application of our prestated theorem will complete the proof for the general quartic. This is easily done, since, for $d=0$, the quartic is symmetric to the y-axis, and the areas cut off are obviously equal.

Also solved by A. Pelletier, F. L. Wilmer, and Paul Wernicke.

3410. [1930, 94]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Construct a triangle given the base, the difference of the base angles, and the bisector of the vertical angle.

Solution by Louise E. Wiederhold, William Smith College.

Let ABC be the required triangle, O its circumcircle, AU the bisector of angle BAC, intersecting BC at U, AD the altitude to BC from A, and AO the circumradius through A. Then the angle DAO equals angle ABC minus angle ACB. Since AU is the bisector of angle BAC, it is also the bisector of the angle DAO. Therefore angle DAU equals $\frac{1}{2}$(angle ABC minus angle ACB). Since angle ABC minus angle ACB is given, $\frac{1}{2}$(angle ABC minus angle ACB) is also known, which together with the length AU determines the right triangle DAU, which can be constructed. The points D, U then determine the line, l, on which BC is to lie. Since the point U on the line l is known, the point U' which is the intersection of the external bisector of angle BAC (perpendicular to AU at A) with the line l can be found. The circle K on UU'as diameter passes through

A. The points B, C are inverse points with respect to this circle since $(U'UBC)$ equals -1. A circle on BC as diameter would therefore be orthogonal to the circle K. But on l there is an infinity of pairs of points which are inverse with respect to K. However there is only one pair of points which are the length BC apart and which (considering AU as the internal bisector of angle BAC) fit the case. In order to find these points B, C it is necessary to construct a circle M orthogonal to K, of radius $\frac{1}{2}BC$. The center of this circle can be found in this manner: construct a right triangle with legs equal to $\frac{1}{2}UU'$ and $\frac{1}{2}BC$. The hypotenuse of this triangle will equal the distance of the center of M from the center of K on line l. (U must be located between K and M). The circle with M as center and $\frac{1}{2}BC$ as radius will cut the line UU' in the required points. B, C, with A determine the triangle.

Also solved by E. J. Buddenbaum, Solomon Kullbak, J. D. Leith, R. K. Morley, William Orange, A. Pelletier, W. A. Rees, Ralph Tocker, Paul Wernicke, and Frank L. Wilmer.

3411. [1930, 94] *Proposed by J. Rosenbaum, Milford, Conn.*

A particle within a closed plane curve is attracted by every point of the curve with a force proportional to the distance from the particle to the point. Find the position of equilibrium of the particle.

Solution by the Proposer.

This problem is a generalization of 3395, November, 1929. There the centers of attraction were finite in number, while here they are all the points of a continuum. Since there, the position of equilibrium is the center of mass of n equal masses placed at the n centers of attraction regardless of the value of n, of the magnitude of the masses, or of the manner of distribution of the centers of attraction, it follows that here, too, the position of equilibrium is the center of mass of the arc of the given curve.

It thus appears that the curve does not need to be a plane curve or closed. The centers of attraction may be even all the points of a given area or volume.

Also solved by A. Pelletier and Paul Wernicke.

NOTES AND NEWS

Professor George Birkhoff has been elected Corresponding member of the Royal Academy of Sciences of the Institute of Bologna.

Dr. M. A. Basoco, of the California Institute of Technology, has been appointed assistant professor of mathematics at the University of Nebraska.

Dr. Theodore Bennett has been appointed assistant professor of mathematics at the University of Wisconsin.

Dr. R. V. Churchill has been promoted to an assistant professorship of mathematics at the University of Michigan.

Assistant Professor A. J. Cook has been promoted to an associate professorship of mathematics at the University of Alberta.

Assistant Professor P. H. Daus, of the University of California at Los Angeles, has been promoted to an associate professorship of mathematics.

Dr. H. A. Davis, of West Virginia University, has been promoted to an assistant professorship of mathematics.

Assistant Professor W. H. Durfee, of Hobart College, has been promoted to a professorship of mathematics.

Dr. C. M. Erikson, of Michigan State Normal College, has been promoted to an assistant professorship of mathematics.

Mr. J. H. Fithian has been promoted to an assistant professorship of mathematics at the Newark College of Engineering.

Mr. H. J. Gay has been promoted to an assistant professorship of mathematics at the Worcester Polytechnic Institute.

Dr. Marion C. Gray has been appointed to a position in the department of development and research of the American Telephone and Telegraph Company.

Dr. E. K. Haviland has been appointed assistant professor of mathematics at Lehigh University.

Dr. G. A. Hedlund has been appointed associate in mathematics at Bryn Mawr College.

Dr. H. M. Hosford has been appointed to a professorship at the University of Arkansas.

Assistant Professor Emma Hyde has been promoted to an associate professorship of mathematics at Kansas State Agricultural College.

Miss Myra I. Johnson has been appointed head of the department of biological science at Whitney Women's College.

Associate Professor R. A. Johnson, of Hunter College, has been appointed associate professor of mathematics at Brooklyn College of the City of New York.

Assistant Professor V. F. Lenzen has been promoted to an associate professorship of physics at the University of California, Berkeley.

Mr. A. J. Lewis has been promoted to an assistant professorship of mathematics at the University of Denver.

Assistant Professor Mayme I. Logsdon has been promoted to an associate professorship of mathematics at the University of Chicago.

Assistant Professor H. F. MacNeish, of the College of the City of New York, has been promoted to an associate professorship of mathematics.

Assistant Professor E. B. Mode has been promoted to a professorship of mathematics at Boston University.

Assistant Professor C. H. Rawlins, of the United States Naval Academy, has been promoted to an associate professorship of mathematics.

Dr. Edward Saibel has been appointed assistant professor of mechanics at the Carnegie Institute of Technology.

Dr. J. A. Shohat has been appointed lecturer in mathematics at the University of Pennsylvania.

Assistant Professor C. A. Shook, of Yale University, has been appointed assistant professor of mathematics at Lehigh University.

Assistant Professor S. P. Shugert has been promoted to a professorship of mathematics at the University of Pennsylvania.

Assistant Professor E. B. Starke, of Rutgers University, has been promoted to an associate professorship of mathematics.

Associate Professor F. C. Touton, of the University of Southern California, has been promoted to be professor of educational research and vice-president of the University.

Dr. Morgan Ward has been promoted to an assistant professorship of mathematics at the California Institute of Technology.

Assistant Professor F. M. Weida, of Lehigh University, has been appointed associate professor of mathematics at George Washington University.

The following appointments to instructorships are announced:
Brooklyn College of the City of New York: Mr. A. W. Landers, Jr.
University of Buffalo: Dr. George C. Munro of the University of Michigan.
Case School of Applied Science: Dr. O. E. Brown.
Lehigh University: Mr. F. S. Beale and Dr. E. H. Cutler.
Los Angeles Junior College: Mr. E. J. Hills.
State Teachers College, Oshkosh: Miss Irene Price.
Princeton University: Mr. A. L. Foster, Mr. J. E. Merrill.
Purdue University: Mr. H. F. S. Jonah.
Radcliffe College: Dr. Mary Curtis Graustein (tutor).
Rutgers University: Mr. H. S. Grant.
Sweet Briar College: Dr. Ethel I. Moody.
South Dakota State College: Mr. Orlin L. Walder and Miss Irene Wente.
United States Naval Academy: Mr. Ernest Hawkins.
Yale University: Mr. Hassler Whitney.

Dr. James M. Taylor, professor emeritus of mathematics at Colgate University, died July 31, 1930 at the age of eighty-six.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TEASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA, Ames, Iowa, May 2-3.
KANSAS, February 15.
KENTUCKY, Lexington, Ky., April 5.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May 10.
MICHIGAN, Ann Arbor, Mich., March 22.
MINNESOTA, Carleton College, May 17.
MISSOURI, Columbia, Mo., November 28.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA, Philadelphia, Pa., Nov. 29.
ROCKY MOUNTAIN, Denver, Colo., April 11-12.
SOUTHEASTERN, Atlanta, Ga., May 2-3.
SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE CUBIC AND BIQUADRATIC EQUATIONS VIETA'S TRANSFORMATION IN THE COMPLEX PLANE

By ARCHIBALD HENDERSON and A. W. HOBBS, University of North Carolina

1. In this paper, a number of methods, possessing a measure of novelty, are presented for the solution of the cubic and biquadratic equations. The transformation attributed to Vieta (1591) presents interesting points if treated as a problem in complex variables. The method employed by Heilermann[1] in the algebraic solution of the biquadratic affords an effective method of solving geometrically the biquadratic equation.

2a. *The Reduced Cubic.* If we replace w in the cubic

$$w^3 + pw + q = 0 \tag{1}$$

by $u+iv$ we obtain, after equating real and imaginary parts separately to zero:

$$u^3 - 3uv^2 + pu + q = 0, \tag{2}$$

$$3u^2v - v + pv = 0. \tag{3}$$

The real intersections of these two curves when considered as complex numbers are the roots of the original equation.

The transformation attributed to Vieta for the solution of the cubic is

$$w = z - (p/3z).$$

If $w = u + iv$ and $z = x + iy$, this transformation represents an inversion in the circle

$$x^2 + y^2 = -p/3,$$

followed by a reflection in the axis of reals, and a translation.

Putting $u+iv$ for w, and $x+iy$ for z we get:

$$u = \frac{x(x^2 + y^2 - \frac{1}{3}p)}{x^2 + y^2}, \qquad v = \frac{y(x^2 + y^2 + \frac{1}{3}p)}{x^2 + y^2}.$$

Applying this transformation to equations (2) and (3) we obtain:

$$(x^3 - 3xy^2)(x^2 + y^2)^3 + q(x^2 + y^2)^3 - p^3(x^3 - 3xy^2)/27 = 0, \tag{4}$$

$$y(x^2 + y^2 + p/3)(x^4 + 2x^2y^2 + y^4 - px^2/3 - py^2/3 + p^2/9)(3x^2 - y^2) = 0. \tag{5}$$

The first of these equations, (4), represents a curve of the ninth degree having a triple point at the origin with distinct tangents:

$$x = 0, \qquad x - y\sqrt{3} = 0, \qquad x + y\sqrt{3} = 0.$$

These lines are also asymptotes of the curve. The only real parts of the second curve are:

$$y = 0, \qquad x\sqrt{3} - y = 0, \qquad x\sqrt{3} + y = 0,$$

[1] Zeitschrift für Mathematik und Physik, vol. 44 (1898).